# HW 8-9

Math 308 D

**Problem 1.** Let

$$A = \begin{bmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{bmatrix}$$

and define

$$W := \{\mathbf{x} \in \mathbb{R}^3 s.t. A\mathbf{x} = \lambda_1 \mathbf{x}\}$$

where $\lambda_1$ is the first (smallest) eigenvalue of $A$.
a) Show that $W$ is a subspace of $\mathbb{R}^3$ and describe the eigenspace of $\lambda_1$.
b) Find an orthonormal basis for $W$.
c) Determine $dim(W)$, the algebraic and geometric multiplicity of $\lambda_1$. Is $A$ defective?
d) Exhibit a basis for the row space of $A$.
e) Find a basis for the column space of $A$ (that is the $\mathcal{R}(A)$).

**Problem 2.** Suppose that $F : \mathbb{R}^3 \to \mathbb{R}^3$, which satisfies $F(\mathbf{e}_1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $F(\mathbf{e}_2) = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ and $F(\mathbf{e}_3) = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are the standard unit basis vectors for $\mathbb{R}^3$.
a) Find a matrix $B$ such that $F(\mathbf{x}) = B\mathbf{x}$.
b) Compute $det(B)$ and $rank(B)$.

**Problem 3.** Let

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 3 \\ 2 & 8 & 12 \end{bmatrix}.$$

a) Is $A$ singular or nonsingular?
b) Calculate $det(A^{20})$.
c) Describe all vectors that are orthogonal to the null space of $A$.

**Problem 4.** a) Find a system of 2 equations and 3 unknowns which is inconsistent.
b) Find a linearly dependent set of 3 vectors in $\mathbb{R}^3$, but each pair of two of them form a linearly independent set of vectors.
c) Compute the $det(A)$ where

$$A = \begin{bmatrix} -1 & 2 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ 2 & 2 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix}.$$

d) Find all values of $\alpha \in \mathbb{R}$, such that

$$\begin{cases} (\alpha - 1)x_1 + (\alpha - 2)x_2 &= 0 \\ 2x_1 + (\alpha - 3)x_2 &= 0 \end{cases}$$

has nontrivial solutions. For each value of $\alpha$ describe the set of the solutions.
e) Find a linear transformation that maps $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

**Problem 5.** Let

$$W = Sp\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\} \quad and \quad V = Sp\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}.$$

a) Prove than $W \cap V$ is a subspace.
b) Find a basis for $W \cap V$.

**Problem 6.** a) Let $A$ be $(n \times n)$ nonsingular matrix. Give at least 10 conditions you know which are equivalent to $A$ being nonsingular.
b) Suppose that $A$ is nonsingular. Prove that for any $(n \times n)$ matrix $B$,

$$det(AB - I) = det(BA - I).$$

**Problem 7.** Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 5 \\ 3 & 6 & 9 & 12 & 15 \end{bmatrix}$$

a) Find the kernel and the nullity of $A$.
b) Find a basis for $\mathcal{R}(A)$.
c) Find a basis for the row space of $A$.
d) Let

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Find the characteristic polynomial of $AB$. Is $AB$ singular or nonsingular?

**Problem 8.** Let

$$A = \begin{bmatrix} 2 & 3 \\ a & b \end{bmatrix}.$$

a) Find $a, b$ such that $A$ has eigenvectors $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\2\end{pmatrix}$.

a) Find a matrix $B$ with eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ and eigenvectors $v_{\lambda_1} = \begin{pmatrix}2\\1\end{pmatrix}$ and $v_{\lambda_2} = \begin{pmatrix}3\\2\end{pmatrix}$ respectively.

**Problem 9.** Let $A$ be an unknown $(3 \times 3)$ matrix and we know that for $\mathbf{b} = \begin{pmatrix}1\\2\\2\end{pmatrix}$, $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}_0 = \begin{pmatrix}1\\3\\5\end{pmatrix}$. In addition $\mathcal{N}(A) = Sp\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$.

a) Describe the set of solutions to $A\mathbf{x} = \mathbf{b}$.
b) Find a basis for $\mathcal{R}(A)$ and determine the $rank(A)$.