Measure of the n-Ball

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This note will derive the following result.

Theorem 1. *Let* $S_n = \{(x_1, x_2, \ldots, x_n) : |x|^2 = \sum_1^n x_j^2 \leq a^2\}$ *be the n-ball of radius a. Denote the measure of this ball by* $\mu_n(a)$*. It satisfies the following recursion:*

$$\begin{aligned}
\mu_n(a) &= \beta_n a^n, \textit{ where} \\
\beta_n &= \left(\frac{2\pi}{n}\right) \beta_{n-2} \\
\beta_0 &= 1 \\
\beta_1 &= 2
\end{aligned}$$

Proof. We define $\mu_0(a) = 1, \mu_1(a) = 2a$. Let $r^2 = x_1^2 + x_2^2 + \ldots x_n^2$ and $\rho^2 = x_{n-1}^2 + x_n^2$.

$$\begin{aligned}
\mu_n(a) &= \int_{r \leq a} d^n x \\
&= \int_{\rho \leq a} \mu_{n-2} \left(\sqrt{a^2 - \rho^2}\right) dx_{n-1} dx_n \\
&= \int_{\rho=0}^{a} \int_{\theta=0}^{2\pi} \beta_{n-2} (a^2 - \rho^2)^{n/2-1} \rho d\rho d\theta \\
&= 2\pi \beta_{n-2} \int_0^a (a^2 - \rho^2)^{n/2-1} \rho d\rho \\
&= \frac{2\pi}{n} \beta_{n-2} (-(a^2 - \rho^2)^{n/2}|_0^a \\
&= \frac{2\pi}{n} \beta_{n-2} a^n \\
&= \beta_n a^n
\end{aligned}$$

□

Corollary 1. *The measure of* $2n$*-balls of radius a is* $\frac{(\pi a^2)^n}{n!}$*. The measure of* $2n+1$*-balls is* $2\frac{(4\pi)^n n!}{(2n+1)!} a^{2n+1}$