Definition of Cosine

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Now that we have a definition of arc length we can define the functions sine and cosine. Let us use the parametrization $(-t, \sqrt{1-t^2}), -1 \leq t \leq 1$ for the the unit semicircle in the upper half plane. The arc length from $(1,0)$ to $(-t, \sqrt{1-t^2})$ is $s(t) = \displaystyle\int_{-1}^{t} \frac{dx}{\sqrt{1-x^2}}$. The traditional notation for arc length on the circle is θ, so we will switch to that notation: $\theta(t) = \int_{-1}^{t} \frac{dx}{\sqrt{1-x^2}}$. Since $\dfrac{d\theta}{dt} > 0$ the inverse function theorem implies that t is a differentiable function of $t(\theta)$. We define $\cos\theta = -t(\theta)$, the x-coordinate of the point on the circle of arc length, θ from $(1,0)$. We define $\sin\theta = \sqrt{1-\cos^2(\theta)}$. Now we have

Theorem 1.

$$fracd \cos\theta d\theta = -\sin\theta$$
$$fracd \sin\theta d\theta = \cos\theta.$$

Proof.

$$\begin{aligned}
\frac{d\cos\theta}{d\theta} &= -\frac{dt}{d\theta} \\
&= -\frac{1}{\frac{d\theta}{dt}} \\
&= -\sqrt{1-t^2} \\
&= -\sin\theta.
\end{aligned}$$

This proves the first equality. The second equality follows from the definition of $\sin\theta$, using the first equality and the chain rule.

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