Implicit Function Theorem

This document contains a proof of the implicit function theorem.

Theorem 1. *Suppose $F(x,y)$ is continuously differentiable in a neighborhood of a point $(a,b) \in \mathbb{R}^n \times \mathbb{R}$ and $F(a,b) = 0$. Suppose that $F_y(a,b) \neq 0$. Then there is $\delta > 0$ and $\epsilon > 0$ and a box $B = \{(x,y) : \|x-a\| < \delta, |y-b| < \epsilon\}$ so that*

1. *For each x such that $\|x-a\| < \delta$ there is a unique y such that $|y-b| < \epsilon$ for which $F(x,y) = 0$. This correspondence defines a function $f(x)$ on $\{\|x-a\| < \delta\}$ such that*

$$F(x,y) = 0 \Leftrightarrow y = f(x) \text{ for } (x,y) \in B.$$

2. *f is continuous.*

3. *f is continuously differentiable and*

$$Df(x) = -\frac{D_x F(x, f(x))}{F_y(x, f(x))},$$

where $Df = [f_{x_1}, \ldots, f_{x_n}]$ and $D_y F = [F_{x_1}, \ldots, F_{x_n}]$.

Proof. 1. Choose $\delta_1 > 0$ and $\epsilon_1 > 0$ so that $F_y(x,y) > 0$ for $\|x-a\| < \delta_1,\ |y-b| < \epsilon_1$. Since $F(a,b) = 0$ and $F(a,y)$ is strictly increasing in y, $F(a, b+\epsilon_1/2) > 0$ and $F(a, b-\epsilon_1/2) < 0$. Let $\epsilon = \epsilon_1/2$ and choose $\delta < \delta_1$ so that $F(x, b+\epsilon) > 0$ and $F(x, b-\epsilon) < 0$ if $\|x-a\| < \delta$. These dimensions define B. For fixed x with $\|x-a\| < \delta$, since $F(x, b-\epsilon) < 0$, $F(x, b+\epsilon) > 0$, and $F(x,y)$ is strictly increasing in y, the intermediate value theorem implies that there is a unique y with $|y-b| < \epsilon$ such that $F(x,y) = 0$. The uniquely determined y defines a function$f(x)$. This proves the first statement.

2. We prove that f is continuous at a. Let $e > 0$ be given. Assume that $e < \epsilon$ Then by the proof of the first statement, there is a $d > 0$ (we may choose $d < \delta$) so that the uniquely defined $f(x)$ in $\{\|x-a\| < d\}$ satisfies $|f(x) - b| < d$. This proves continuity at a. We can repeat this argument at any point $(a_1, f(a_1)) \in B$, proving that f is continuous on $\{\|x-a\| < \epsilon\}$.

3. By differentiability

$$\begin{aligned} 0 = F(x, f(x)) &= F(a,b) + \sum_j P_j(x, f(x))(x - a_j) + Q(x, f(x))(f(x) - f(a)) \\ &= \sum_j P_j(x, f(x))(x - a_j) + Q(x, f(x))(f(x) - f(a)), \end{aligned}$$

where $P_j(x, f(x)), Q(x, f(x))$ are continuous at a. Rewrite this as

$$Q(x, f(x))(f(x) - f(a)) = -\sum_j P_j(x, f(x))(x - a_j).$$

Since $Q(x, f(x))$ is continuous at a and $Q(a, (f(a)) = f_y(a, b) > 0$, $Q(x, f(x)) > 0$ for x near a and we can divide by it to get

$$f(x) = f(a) + -\sum_j \frac{P_j(x, f(x))}{Q(x, f(x))}(x - a_j).$$

Each term $\frac{P_j(x, f(x))}{Q(x, f(x))}$ is continuous at a so f is differentiable at a. Moreover

$$f_{x_j}(a, b) = -\frac{F_j(a, b)}{F_y(a, b)}.$$

You might like this bad notation:

$$\frac{\partial y}{\partial x_j} = -\frac{\partial F_{x_j}}{\partial F_y}.$$

□