# Baire Category Theory

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This document will be evolving (like a blog) over time. It will include an introduction to Baire category theory and applications. To be explicit I will assume we are working in $\mathbb{R}$, but many of the statements and proofs will be correct in a more general context. The concept of countability is crucial in the definitions. I will start off with the definitions. Their meaning will be developed in the applications. There are many good references. I will often take proofs from John Oxtoby's book, *Measure and Category*, [1].

**Definition 1.** *A set $E$ is **dense** in an interval $I$ if for every subinterval $J \subset I$ , $J \cap E \neq \varnothing$. A set $E$ is **nowhere dense** if it is not dense in any interval.*

**Remark 1.** *If $E$ is nowhere dense, then for every interval $I$ there is some subinterval $J$ of $I$ such that $J \cap E = \varnothing$. This is equivalent to the statement that the closure of $E$, $\overline{E}$ has no interior. $E$ is nowhere dense if and only if $\overline{E}$ is nowhere dense. If $E$ is nowhere dense every open set must contain points in $\overline{E}^c$, so the complement of the closure of a nowhere dense set is dense. If the complement of the closure of $E$ is dense then the $E$ is nowhere dense.*

Suppose $E_1$ is nowhere dense. Then $\overline{E_1} \neq \mathbb{R}$, so there is a point $p \notin \overline{E_1}$ and hence a nonempty compact ball around $p$, $B_1$, that is disjoint from $E_1$. We can suppose the radius of $B_1$ is less than 1. Now take a second nowhere dense set $E_2$. Then $B_1$ is not contained in $\overline{E_2}$ so we can select a point in the interior of $B_1$ and a closed ball $B_2$ of radius less than $1/2$ inside $B_1$ that is disjoint from $E_2$. This ball will also be disjoint from $E_1$. Let $E_j$ be a sequence of nowhere dense sets. We can construct a nested sequence of balls $B_{j+1} \subset B_j$ which are disjoint from $\cup E_j$ and by the nested interval theorem they have a non-empty intersection $\cap B_j$. We have proved that $\cup E_j \neq \mathbb{R}$. This is the Baire category theorem which we will now state.

**Definition 2.** *A set is of **first category** if it is a **countable** union of nowhere dense sets. For example $\mathbb{Q}$ is a set of first category. A set is of **second category** if it is not a set of first category. A **generic** set is the complement of a set of first category.*

**Theorem 1 (Baire Category Theorem).** $\mathbb{R}$ *is not of first category. If $C$ is a set of first category then there are points not in $C$. A set of first category can not be all of $\mathbb{R}$. A generic set in $\mathbb{R}$ is dense.*

*Proof.* The only thing left to prove is the density statement. An examination of the proof shows that any open set contains a point not in $\overline{E_1}$ and hence a nonempty open set disjoint from $\overline{E_1}$. Continuing the proof we conclude that any open set contains a point of $(\cup E_j)^c$. In other words any neighborhood of any point contains a point of $(\cup E_j)^c$. This implies that $(\cup E_j)^c$ is dense. This also proves that $(\cup \overline{E_j})^c$ is dense. $\square$

**Corollary 1.** *The intersection of a dense sequence of open sets is dense.*

*Proof.* Let $G_j$ be open and $\overline{G_j} = \mathbb{R}$. Let $F_j = \mathbb{R} - G_j$. Then $F_j$ is nowhere dense. It follows that $\cap_j G_j = (\cup_j F_j)^c$ is a generic set and hence dense. $\square$

**Remark 2.** *The same proof shows that any interval (open, closed, or neither) is not a set of first category and it also proves that $\mathbb{R}^n$ is not of first category.*

We will use this result to prove some striking theorems. We also will use the following definitions.

**Definition 3.** *A subset $E$ of $\mathbb{R}^n$ is a $G_\delta$ if $E = \cap_{n=1}^\infty G_n$ where each $G_n$ is and open set. Notice the intersection is* <span style="color:red">countable</span>. *A set $B$ is an $F_\sigma$ it is a* <span style="color:red">countable</span> *union $B = \cup F_n$ where each $F_n$ is closed.*

**Proposition 1.** *Closed and open sets are both $G_\delta$'s and $F_\sigma$'s.*

*Proof.* It is clear that an open set is a $G_\delta$. Every open set is a countable union of closed balls with rational centers and rational radii. Since the complement of an $F_\sigma$ is a $G_\delta$ the result for closed sets follows. $\qquad\square$

**Proposition 2.** *The set $\mathbb{Q}$ of rational numbers is an $F_\sigma$ but not a $G_\delta$.*

*Proof.* If $\mathbb{Q}$ were a $G_\delta$, then the complement of $\mathbb{Q}$ is the countable union of closed sets $\mathbb{Q}^c = \cup G_n^c$. Since $\mathbb{Q}^c$ has no interior, $(G_n^c)^o = \varnothing$, where $E^o$ denotes the interior of a set. But we have written $\mathbb{R} = \mathbb{Q} \cup (\cup G_n^c)$ as a countable union of nowhere dense sets and this contradicts the Baire category theorem. $\qquad\square$

Let $f$ be a function defined on $\mathbb{R}^n$. We define the oscillation of $f$ on a set $W$ by $o_W(f) = \sup_W(f) - \inf_W(f)$, where $\sup_W(f)$, $\inf_W(f)$ are the sup and inf of $f$ on $W$. Let $B_n(x) = \{y : \|y - x\| < 1/n$ and define $o_f(x) = \lim_{n \to 0} o_{B_n(x)}(f)$.

**Proposition 3.** *$f$ is continuous at $a$ if and only if $o_f(a) = 0$.*

*Proof.* This is true by the $\epsilon - \delta$ definition of continuity. $\qquad\square$

**Proposition 4.** *For every number $r$, $\{x | o_f(x) < r\}$ is an open set (and $\{x | o_f(x) \geq r\}$ is closed).*

*Proof.* If $o_f(x) < r$ then for some $n$, $o_{B_n(x)}(f) < r$ and then for all $y \in B_n(x)$, $o_f(y) < r$. $\qquad\square$

**Theorem 2.** *Let $D(f)$ be the set of discontinuity points of $f$. Then $D(f) = \{x | \cup_k \{o_f(x) \geq 1/k\}$ and hence $D(f)$ is an $F_\sigma$.*

**Theorem 3.** *Let $F = \cup_{n=1}^\infty F_n$ be an $F_\sigma$, where each $F_n$ is closed. Then there is a function $f$ with $D(f) = F$.*

*Proof.* (Taken from [1]. ) We will assume $F \neq \mathbb{R}^n$. Let $A_n$ be the set of rational points in $F_n^o$ (might be empty) and let
$f_n = \chi_{F_n \setminus A_n} = \chi_{F_n} - \chi_{A_n}$. Then

$$o_{f_n}(x) = \begin{cases} 1 \text{ if } x \in F_n \\ 0 \text{ if } x \notin F_n. \end{cases}$$

It is important that $F_n$ be closed for this to be correct. This implies $D(f_n) = F_n$. Now let $a_n = 1/n!$ and $f = \sum_1^\infty a_n f_n$. We claim that $a_n > \sum_{m>n} a_m$. This is an easy thing to check. By the Weierstrass $M$-test, the series converges uniformly. Since each $f_n$ is continuous on $\mathbb{R} - F$, $f$ is continuous on $\mathbb{R} - F$. Now we estimate the oscillation of $f$ at each point of $F$. On $F_1$

$$0 \leq \sum_2^\infty a_m f_m(x) \leq \sum_2^\infty a_m < a_1.$$

In each neighborhood of a point $x$ of $F_1^o$ the maximum of $f$ is at least $a_1$ and the minimum is less than or equal to $\sum_2^\infty a_m$. So $o_f(x) \geq a_1 - \sum_{m \geq 2} a_m > 0$. If $x \in \partial F_1$ then $f_1(x) = 1$ and the maximum of $f$ is at least $a_1$. Again, the minimum is no more than $\sum_2^\infty a_m$, so $o_f(x) > 0$. A similar argument works at each point of $F_n - F_{n-1}$ to prove that $o_f(x) > 0, x \in F$, thus $D(f) = F$. $\qquad\square$

Now let's see what we can say about limits of continuous functions.

**Theorem 4.** *Suppose $f_n$ is continuous on $I$ for each $n$ and suppose $\lim_{n\to\infty} f_n(x) = f(x), x \in I$, where $I$ is an interval. Then $D(f)$ is a set of first category.*

*Proof.* Let $c > 0$. First, we prove that $E_c = \{x : o_f(x) \geq c\}$ is nowhere dense. If we know this, then $D(f) = \cup_k \{x | o_j(x) \geq 1/k\}$ is of first category.

We will show that any compact subinterval in $I$ has a subinterval that is not a subset of $E_c$. Choose $d$ with $3d < c$, let $K$ be a compact subinterval of $I$, and let

$$F_n = \cap_{k,l \geq n} \{x \in K : |f_k(x) - f_l(x)| \leq d\}.$$

$F_n$ is the intersection of closed sets and hence closed. Convergence on $I$ and the Cauchy criterion implies that $\cup_n F_n = K$. By the Baire category theorem, not every $F_n$ can be nowhere dense. Then for some $n$ there is a compact interval $J$ such that $J \subset F_n$. Now fix $l = n$ and let $k \to \infty$ to get $|f(x) - f_n(x)| \leq d$ for all $x \in J$. Since $f_n$ is continuous there is a $\delta > 0$ so that $|f_n(x) - f_n(y)| < d$ for $|x - y| < \delta$ (uniform continuity of $f_n$ on $J$). Then $|f(x) - f(y)| < 3d$ when $|x - y| < \delta$. So if $x \in J$, $o_f(x) < 3d$ and hence $J \not\subset \{x : o_f(x) \geq c\}$. $\qquad\square$

**Corollary 2.** *The set of discontiuities of a derivative is a set of first category. The set of points of continuity of a derivative is a generic set and hence dense. There are "lots" of points of continuity of a derivative.*

*Proof.* If $f$ has a derivative, $f$ is continuous. Then

$$f'(x) = \lim_{n\to\infty} \frac{f(x + 1/n)}{1/n},$$

so $f'$ is a limit of a sequence of continuous functions. We have to be more careful on an interval since if $f$ is defined on an interval $I$, $f(x + 1/n)$ might not be defined on that interval. If $f$ has a derivative on a closed interval, we can extend $f$ linearly beyond the endpoints to a function that is differentiable everywhere. If $I$ is not closed, we can restrict $f$ to a sequence of closed sub intervals that exhaust $I$ and conclude that the set $D(f')$ is a countable union of sets of first category and hence is of first category. $\qquad\square$

# References

[1] John Oxtoby, *Measure and Category*, Springer-Verlag, 1996.