Math 335 Sample Problems

One notebook sized page of notes will be allowed on the test. The test will cover through §7.6

1. Suppose $\sum_0^\infty a_n$ converges. Prove that $\sum_0^\infty \frac{a_n}{n+1}$ converges and

$$\int_0^1 \sum_0^\infty a_n x^n dx = \sum_0^\infty \frac{a_n}{n+1}.$$

2. (a) Suppose f_n converges uniformly on S. Prove that $|f_n|$ converges uniformly on S.
   (b) Suppose f_n is Riemann integrable on $I \subset \mathbb{R}$. Assume that f_n converges uniformly on I to f. Prove that
   $$\lim_{n\to\infty} \int_I f_n^2 = \int_I f^2.$$
   (c) Suppose f_n converges uniformly on S. Does f_n^2 converge uniformly on S? Give a proof or counterexample.

3. Using power expansions of elementary transcendental functions prove that

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = 1.$$

4. Prove that

$$\frac{1}{n!} > \sum_{j=n+1}^\infty \frac{1}{j!},$$

for $n \geq 1$.

5. Suppose that $a_n \geq 0$ and $\displaystyle\sum_{n=0}^\infty a_n$ diverges; and suppose that $\displaystyle\sum_{n=0}^\infty a_n x^n$ converges for $|x| < 1$. Prove

$$\lim_{x\to 1^-} \sum_{n=0}^\infty a_n x^n = +\infty.$$

6. Suppose f_n is a sequence of continuous functions that converges uniformly on a set W. Let p_n be a sequence of points in W that converges to a point $p \in W$. Prove that $\lim_{n\to\infty} f_n(p_n) = f(p)$.

7. Let be a sequence of continuous functions in $I = [a, b]$ and suppose $f_n(x) \geq f_{n+1}(x) \geq 0$ for all $x \in I$. Suppose $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in I$ (point-wise convergence to 0). Is the convergence uniform? Give a proof or a counterexample.

8. Prove that $\displaystyle\sum_{n=0}^{\infty} \frac{x}{(1+|x|)^n}$ converges for all x, but the convergence is not uniform.

9. Assume $p \geq 1,\ q \geq 1$. Prove that
$$\int_0^1 \frac{t^{p-1}}{1+t^q} dt = \frac{1}{p} - \frac{1}{p+q} + \frac{1}{p+2q} \cdots .$$
Give careful justification of any manipulations.

10. Suppose $a_n > b_n > 0$, $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$. Does $\displaystyle\sum_1^{\infty} (-1)^n b_n$ converge? Give a proof or a counterexample.

11. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ converges uniformly for $x \in [a, b], 0 < a < b < 2\pi$, but does not converge absolutely for any x.

12. Prove that
$$\int_0^1 \left(\frac{\log(1/t)}{t}\right)^{1/2} dt = \sqrt{2\pi}.$$

13. Prove that $\displaystyle\sum_1^{\infty} (-1)^n \frac{\sin nx}{n}$ converges uniformly on $\{|x| < 1\}$ to a continuous function.

14. Folland §7.5, #9.

15. Let f_n be a sequence of functions defined on the open interval (a, b). Suppose $\lim_{x\to a^+} f_n(x) = a_n$ for all n. Suppose $\displaystyle\sum_1^{\infty} f_n$ converges uniformly on (a, b) to a function f. Prove that $\displaystyle\sum_1^{\infty} a_n$ converges and $\displaystyle\lim_{x\to a^+} f(x) = \sum_1^{\infty} a_n$. Do not assume f_n is continuous on (a, b).

16. Folland, §7.5, #14.

17. Suppose the series $\sum_1^{\infty} a_n$ converges. Prove that $\sum_1^{\infty} \frac{a_n}{n^x}$ converges for $x \geq 0$. Let $f(x) = \sum_1^{\infty} \frac{a_n}{n^x}$. Prove that $\lim_{x\to 0^+} f(x) = \sum_1^{\infty} a_n$.

18. Problem #13, §7.5 of Folland.

19. Let $p_j(t) = e^{-t}\dfrac{t^j}{j!}$.

    (a) Suppose $\sum_0^\infty a_n$ converges. Let $s_n = \sum_0^n a_j$. Prove that

    $$\lim_{t\to\infty} \sum_0^\infty s_j p_j(t) = \sum_0^\infty a_n.$$

    (b) Compute this limit in the case that $a_n = x^n$ for those x for which the limit exists (even in the case that $\sum x^n$ does not converge). This limit is called the Borel regularized value. What does this give for the *Borel regularized value* of $1 - 2 + 4 - 8 + 16 \pm \dots$?

20. Prove that

    $$\int_0^1 \frac{\log(x)}{x-1} = \sum_{k=1}^\infty \frac{1}{k^2}.$$

    The integral is improper. Write the integrand as a series, integrate term-by-term and use Abel's theorem.

21. You will need to know the definitions of the following terms and statements of the following theorems.

    (a) Abel's theorem
    (b) Uniform convergence of a sequence or series of functions
    (c) Weierstrass M-test
    (d) Continuity of a uniform limit of continuous functions
    (e) Integration and differentiation of a sequence or series
    (f) Power series
    (g) Radius of convergence of a power series
    (h) Integration and differentiation of a power series
    (i) Improper integrals dependent on a parameter
    (j) Uniform convergence of an improper integral
    (k) Integration and differentiation of an improper integral
    (l) Gamma function

22. There may be homework problems or example problems from the text on the midterm.