Axiom of Choice, Trichotomy, and The Continuum Hypothesis

Martha A. Wagley
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1 Introduction

Leonard Gillman, in his paper *Classical Surprises Concerning the Axiom of Choice and the Continuum Hypothesis*, published by the American Mathematical Monthly, introduced how Trichotomy and the Continuum Hypothesis imply the Axiom of Choice. In this paper we will outline Gillman's proofs for the Axiom of Choice and discuss why they are important in Set Theory and beyond. We will define and outline what Trichotomy, the Continuum Hypothesis, and the Axiom of Choice are and then show how they relate to each other. Interestingly, both Trichotomy and the Continuum Hypothesis seem completely isolated from the Axiom of Choice at first glance, but in the early 20th century it was discovered that they both imply the Axiom of Choice. This discovery allowed for the generalization of the Axiom of Choice and widened its applications in Mathematics.

2 Definitions

The next few sections are devoted to deriving the Axiom of Choice from Trichotomy and the Continuum Hypothesis. The following definitions outline the vocabulary needed to understand the derivations.

- A **cardinal number** of a set is the number of elements in a set. An infinite set is said to have infinite cardinal while a finite set is said to have finite cardinal
- The Hebrew letter **aleph** is the cardinal number of an infinite well ordered set and is denoted by $\aleph$
- A **well ordered** set is one that has a least element

The above definitions are used in the following theorems explaining the relationship between Trichotomy, the Continuum hypothesis, and the Axiom of Choice.

3 The Axiom of Choice

3.1 Introduction

The Axiom of Choice was formulated by Ernst Zermelo in 1904 and met with much controversy in its early years. It was thought to be dependent on the Zermelo-Fraenkel axioms, which were the foundations of a branch of set theory, but the discovery that the Axiom of Choice follows from Trichotomy and the Continuum Hypothesis freed the Axiom of Choice from dependency.

The Axiom of Choice asserts that for every collection of nonempty sets there is one set contained in the collection that contains an element from each set in the collection. A formal statement of the Axiom of choice follows:

Theorem 3.1. *For every collection of nonempty sets S, there exists a function F such that F(S) is a member of S for every possible S*

A more general and non rigorous statement of the Axiom of Choice is that if we have an infinite number of parking places in a lot and each parking place contains one vehicle. Then it is possible to pick one vehicle from each parking place without knowing which vehicle to select from each parking place.

The Axiom of Choice is used by many Mathematicians, but is rarely recognized as a formal statement. In the next two sections we will present two proofs in which the Axiom of Choice is formalized.

4 Axiom of Choice and the Well Ordering Theorem

An important application of the Axiom of Choice is the Well Ordering Theorem, which states that every set can be well ordered. Interestingly, the Axiom of Choice is an easy consequence of the Well Ordering Theorem and both are equivalent to the statement that every infinite cardinal is an aleph. The following theorems and proofs show this concretely.

Theorem 4.1. *The Well Ordering Theorem implies the Axiom of Choice.*

Proof. Let S be a collection of nonempty sets. Let A be a well ordered set in S. The idea of the proof is that the choice function sends each well ordered subset of S to its least element. □

5 Trichotomy

5.1 Introduction

The Law of Trichotomy states that any two cardinals a and b satisfy exactly one of the conditions $a < b$, $a = b$, or $a > b$. In more general terms, the Law of Trichotomy guarantees that every cardinal is either positive, negative, or zero. In 1915 Friedrich Hartogs proved that Trichotomy implies the Axiom of Choice. The following proof of Hartogs discovery uses his methods and the outline in Leonard Gillman's paper.

5.2 Hartogs-Sierpinski Theorem

The proceeding theorem and the theorems proceeding it are stated as they are stated in Leonard Gillman's paper:

Theorem 5.1. *The cardinal number of a set is the number of elements in a set. To each infinite cardinal m is associated an aleph $\aleph(m)$ satisfying the relations*

$$\aleph(m) \not\leq m$$

and

$$\aleph(m) \leq 2^{2^{2^m}}$$

Proof. Let m be an infinite cardinal and M be a set of cardinal m. Since by definition 2^M is a subset of M and a subset N of 2^M is a set of subsets of M. By the properties of sets the elements of N are well ordered. Let W be the set of all subsets N of 2^M. Since W is a set of subsets of 2^M, $W \subset 2^{2^M}$.

The proof of Theorem 4.1 continues in this way and concludes that $\aleph(\mathrm{m}) \leq 2^{2^{2^m}}$. It is helpful both in the relationship between Trichotomy and the Axiom of Choice and in the relationship between the Continuum Hypothesis and the Axiom of Choice.

□

Corollary 5.2. *No cardinal is greater than all the alephs.*

The previous two statements make up Trichotomy and in the next section we will prove that the two together are equivalent to the Axiom of Choice

5.3 Proof that Trichotomy is equivalent to the Axiom of Choice

Theorem 5.3. *The Law of Trichotomy is equivalent to the Axiom of Choice.*

Proof. We know that the Axiom of Choice is equivalent to the statement that every infinite cardinal is an aleph. Therefore we need to reduce the Law of Trichotomy to the statement that every infinite cardinal is an aleph.

We want to prove the following theorem using Theorem 4.1 and Corollary 4.2. Corollary 4.2 ensures that any two alephs are comparable which implies that any two cardinals are comparable. Trichotomy is the statement that any two cardinals are comparable. Now we can assume Trichotomy and consider any infinite cardinal m. By Theorem 4.1, $\aleph(m) \nleq m$. By Trichotomy, $\aleph(m) > m$. This inequality implies that m is strictly less than some aleph and is therefore an aleph. Therefore, all infinite cardinals are alephs. Since the Axiom of Choice and the Well Ordering Theorem are equivalent and the Well Ordering Theorem is equivalent to the statement that every infinite cardinal is an aleph, the Axiom of Choice is equivalent to Trichotomy.

□

6 The Continuum Hypothesis and the Axiom of Choice

6.1 Introduction

The Continuum Hypothesis states that for every infinite cardinal m, there is no cardinal n satisfying $m < n < 2^m$. Sierpinski was the first mathematician to prove that the Continuum Hypothesis implies the Axiom of Choice. The following theorems and lemmas outline the proof of this implication.

6.2 Proof

We want to show the following theorem using the fact that the Axiom of Choice is equivalent to the statement that every infinite cardinal is an aleph.

Theorem 6.1. *The Continuum Hypothesis implies the Axiom of Choice*

Proof. The following lemmas are necessary to prove Theorem 5.1

Lemma 6.2. *If $p \geq \aleph_0$, then $2^p + p = 2 \cdot 2^p = 2^p$ where p is a cardinal*

Proof. We know from the properties of the addition of cardinals that $1 + p = p$ if $p \geq \aleph_0$. Therefore,

$$2^p \leq 2^p + p \leq 2^p + 2^p = 2 \cdot 2^p = 2^{1+p} = 2^p$$

From the above inequality we get that $2^p \leq 2^p + p \leq 2^p$, $2^p + p = 2^p$. Therefore, $2^p + p = 2 \cdot 2^p = 2^p$, which is what we were trying to show.

□

Lemma 6.3. *If a and p are cardinals satisfying 2p = p and a + p = 2^p, then $a \geq 2^p$*

Proof. Let P and P' be disjoint sets of power p, and let A be a set of power a disjoint from P. Then

$$|A \cup P| = a + p$$

The previous inequality holds because $A \cap B = \oslash$, A and B' are representative sets of a and b, and $|A \cup B| = a + b$ is the definition of the addition of cardinals when the conditions on A, B, a, and b are satisfied. Also,

$$a + p = 2^p = 2^{p+p}$$

The first inequality is assumed in the theorem and the second equality holds because p is a cardinal that satisfies $2p = 2$, which implies that p is an infinite cardinal. An infinite cardinal is an aleph and the definition of addition of alephs is $\aleph_0 + \aleph_0 = \aleph_0$. This implies that $p + p = p$. Also,

$$2^{p+p} = |2^{P \cup P'}|$$

The previous inequality holds because $P \cap P' = \oslash$, P and P' are representative sets of p, and $|P \cup P'| = p + p$ is the definition of the addition of cardinals when the conditions on P, P', and p are satisfied.

We will now use the above equalities to prove Lemma 5.3. Let f be an one to one mapping of $A \cup P$ onto $2^{p \cup P'}$. Let W be a subset of P' and let W^* be the set W and the elements x of P that do not belong to the set $f(x)$. Therefore E^* is a subset of $|P \cup P'|$, and for all x in P, x is an element of E^*. It follows that $E^* = f(y)$ for some y in A. E is any one of the 2^p subsets of P', and the correspondence between E and E^* is one-to-one. Therefore there are 2^p sets of the form E^*, hence there are 2 corresponding elements y in A. Consequently, A has at least 2^p elements. Thus $a \leq 2^p$. □

Lemma 6.4. *For n = 1,2, or 3, if*

$$\aleph(m) \leq p_n,$$

then either m is an aleph or

$$\aleph(m) \leq p_{n-1}.$$

Proof. We will use the following notations in the proof:

$$p_0 = m$$

$$p_1 = 2^{p_0} = 2^m$$

$$p_2 = 2^{p_1} = 2^{2^m}$$

$$p_3 = 2^{p_2} = 2^{2^{2^m}}$$

These p_n statisfy $p_n \leq \aleph_0$ which means by Lemma 5.2 that

$$2 \cdot 2^{p_n} = 2^{p_n}.$$

Using the above notation and that $\aleph(\mathrm{m}) \leq p_n$ we can see that

$$p_{n-1} \leq \aleph(m) + p_{n-1} \leq p_n + p_n = p_n = 2^{p_{n-1}}.$$

According to the above inequality, $p_{n-1} \leq \aleph(m) + p_{n-1} \leq p_{n-1}$. The Continuum Hypothesis states that for every infinite cardinal m, $m < 2^m$, and there is no cardinal n satisfying $m < n < 2^m$. Since p_{n-1} is an infinite cardinal and $\aleph(m) + p_{n-1}$ is also a cardinal and $p_{n-1} < 2^{n-1}$ the inequality on the left of the above expression must be a strict inequality and the inequality on the right must be an equality. The expression becomes either:

$$p_{n-1} < \aleph(m) + p_{n-1} = 2^{n-1}$$

or

$$p_{n-1} = \aleph(m) + p_{n-1} < 2^{n-1}$$

If $\aleph(m) + p_{n-1} = 2^{n-1}$, by Lemma 5.3, which states that if $2p = p$ and $a + p = 2^p$, then $a \geq 2^p$, with $p = p_{n-1}$ and $a = \aleph(m)$ we get the following inequality

$$\aleph(m) \geq 2^{n-1} \geq m$$

Whence, m is an aleph. If, on the other hand, $p_{n-1} = \aleph(m) + p_{n-1}$, the following inequality is true:

$$\aleph(m) \leq p_{n-1}$$

Therefore, either m is an aleph or $\aleph(m) \leq p_{n-1}$ and Lemma 5.4 follows

□

Theorem 4.1 tells us that $\aleph(m) \leq 2^{2^{2^m}} = p_3$. We know from Lemma 5.4 that the inequality $\aleph(\mathrm{m}) \leq p_{n-1}$ holds for p_3. By Lemma 5.4 either m is an aleph or $\aleph(\mathrm{m}) \leq p_{n-1}$ holds for $n = 2$. This implies that either m is an aleph or $\aleph(\mathrm{m}) \leq p_{n-1}$ holds for $n = 1$. This implies that either m is an aleph or $\aleph(\mathrm{m}) \leq p_{n-1}$ holds for $n = 0$. We know that $\aleph(\mathrm{m}) \leq p_{n-1}$ does not hold for n = 0 so m must be an aleph. The fact that m is an aleph is synonymous with the statement that every infinite cardinal is an alpeh. Therefore Theorem 5.1 holds and the Continuum Hypothesis implies the Axiom of Choice. □

7 Conclusion

The discovery that the Axiom of Choice follows from both Trichotomy and the Continuum Hypothesis ensured that the Axiom of Choice is independent of the Zermelo-Fraenkel axioms and set theory as a whole. This allows the Axiom of Choice to be applied to all kinds of disciplines. The Axiom of Choice is used in Set Theory to validate comparing cardinalities of sets. One of the most controversial facts that comparing cardinalities of sets implies is that different infinities have different sizes and that their sizes can be compared. Philosophers frequently find this application troubling because it implies something about infinity, which many people believe is nonexistent and contradictory. The problem is that the Axiom of Choice is necessary for many proofs and theorems in the field of Mathematics. It must be assumed for many theorems to make any sense at all. The Axiom of Choice is generally accepted in Mathematics, but there is still some controversy over its implications.

References

[1] Leonard Gillman. Two Classical Surprises Concerning the Axiom of Choice and the Continuum Hypothesis. *American Mathematical Monthly*, 109:pp.544–553, 2002.