A (Less than Practical) Solution to the N-Body Problem

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1 Introduction

The classical N-body problem is simply-posed, yet it has persisted as an active source of research for mathematicians and astronomers for hundreds of years. Given N particles which interact with one another only via Newtonian inverse-square gravity, and some intitial configuration, the problem asks for the time-evolution of each particle's position and velocity. In the $N = 2$ case, a solution is readily found. The brunt of the research on the general problem today is in the realm of computational physics, which may contribute to the popular notion that the problem is not analytically soluble for $N > 2$. But as Donald G. Saari points out [5], the problem in its canonical form was solved under extrememely general conditions for $N = 3$ by mathematician-astronomer Karl Sundman in 1907-12 (culminating in [6]). This solution was extended to the (almost entirely) general N-body problem by Wang Qiu-Dong in 1991 [3].

In this paper, our main goal (as in [5]) is to explore Sundman's solution to the 3-body problem. The result was published (in French) in three very large papers, so our coverage will neccessarily be simplistic. We begin by presenting the basic problem and the 2-body case; then we will chart Sundman's proof, using our knowledge of complex variables to gain some understanding of the method.

1.1 Statement and the 1-body problem

Suppose we have N point particles allowed to move throughout $\mathbb{R}^3$. Denote the i^{th} particle's mass by m_i and its position by $\mathbf{r}_i$. To be physically exact, we will scale the mass by the gravitational constant G so that we may set it to unity. We will use the dot notation, e.g. in $\dot{\mathbf{r}}_i = \mathbf{v}_i$, exclusively to mean the derivative with respect to time t (in contrast to $\mathbf{r}_i'$, for instance). By r_{jk} we mean the distance $|\mathbf{r}_j - \mathbf{r}_k|$. We define the interaction between the particles as follows:

Definition: The *Newtonian inverse square law* for gravity is such that the magnitude of force between particles j and $k \neq j$ is given by $\frac{m_j m_k}{r_{jk}^2}$, and acts on j in the direction of k. So by the Newtonian definition of force and vector superposition, the equation of motion for the j^{th} particle is

$$m_j \ddot{\mathbf{r}}_j = \sum_{k \neq j} \frac{m_j m_k (\mathbf{r}_k - \mathbf{r}_j)}{r_{jk}^3}, \quad j = 1, ..., N. \tag{1}$$

The problem, as stated in the famous contest of *Acta Mathematica* in 1885 [2], is to find a series expression for the coordinates of each $\mathbf{r}_i$ in a variable which is a known function of time, and for all values of which the series converges uniformly. The original statement also allowed us to assume that no two particles ever collide, but since this is unrealistic (and collisions are interesting), we remain general for now.

We will need some basic physics results in our work. Namely, we would like to understand the solutions of the central or 1-body force problem: this is the 2-body problem in which one of the particles is fixed in position, say at the origin. For proof, see standard classical mechanics texts (e.g. [1]). Note that in this case we can identify a fixed 'orbital plane' in $\mathbb{R}^3$ to which the orbit is restricted because of the fact that angular momentum is conserved (see (4) below).

Theorem 1. *The equation of motion for the 1-body problem with one particle fixed at the origin and the other at* $\mathbf{r}$ *may be written*

$$\ddot{\mathbf{r}} = -\frac{\mathbf{r}}{r^3}$$

where $r = |\mathbf{r}|$ *and the masses and gravitational constant have been absorbed into* $\mathbf{r}$*. The solutions to this differential equation may be written in polar coordinates in the plane defined by the particle's motion as*

$$r(\theta) = \frac{a}{1 - \epsilon\cos(\theta)} \tag{2}$$

where a *is a positive constant, and* ϵ *is a positive constant called the* eccentricity*. For* $0 \leq \epsilon < 1$*, the orbit this equation describes is an ellipse; for* $\epsilon = 1$*, it is a parabola; and* $\epsilon > 1$*, a hyperbola.*

1.2 First integrals

We can use the symmetry of the mutual force law (1) to find what we call first integrals of the system. They form the basis of one possible route to a solution in initial value problems like this one.

Definition: A *first integral* of a system of differential equations is a function which remains constant along any solution of the system, its value dependent on the particular solution. The integral is *algebraic* if it can be solved algebraically for one variable in terms of the others.

We'd like to find as many of these integrals as we can. Note that in the language of differential equations we have $6N$ dependent variables here: for each particle, the 3 coordinates of $\mathbf{r}_j$ and the 3 components of $\mathbf{v}_j = \dot{\mathbf{r}}_j$. First integrals can effectively decrease this number. First we sum (1) over all particles, to form $\sum_j m_j\ddot{\mathbf{r}}_j$. For each term $\frac{m_j m_k(\mathbf{r}_k - \mathbf{r}_j)}{r_{jk}^3}$ in this sum, there is an antisymmetric term $\frac{m_k m_j(\mathbf{r}_j - \mathbf{r}_k)}{r_{kj}^3}$. Thus, the sum must be zero: $\sum_{j=1}^N m_j\ddot{\mathbf{r}}_j = 0$. By integrating this finite sum twice and setting the integration constants to $\mathbf{0}$ (corresponding to shifting our coordinate system either in space or by a 'boost' of relative constant velocity), we obtain 6 first integrals (one for each component of the two vector equations that follow):

$$\sum_{j=1}^{N} m_j\mathbf{v}_j = \sum_{j=1}^{N} m_j\mathbf{r}_j = \mathbf{0}. \tag{3}$$

Now take the cross product $\mathbf{r}_j \times (m_j\ddot{\mathbf{r}}_j)$, and sum this over all j. The product distributes over all terms of (1). The terms containing $\mathbf{r}_j \times \mathbf{r}_j = 0$ disappear; and for each term $\frac{m_j m_k(\mathbf{r}_j \times \mathbf{r}_k)}{r_{jk}^3}$ in the remaining sum, there is an identical term with j, k reversed which cancels with this one by the anticommutativity of the cross product. So we have

$$\sum_{j=1}^{N} m_j(\mathbf{r}_j \times \ddot{\mathbf{r}}_j) = 0 = \frac{d}{dt}\sum_{j=1}^{N} m_j(\mathbf{r}_j \times \mathbf{v}_j)$$

by the product rule for the cross product. So we have, for some constant $\mathbf{c}$ which we call the system *angular momentum*,

$$\sum_{j=1}^{N} m_j(\mathbf{r}_j \times \mathbf{v}_j) = \mathbf{c}. \tag{4}$$

Now define the *self-potential* U by

$$U = \sum_{j<k} \frac{m_j m_k}{r_{jk}}$$

which is the negative of what is known in physics as the system's gravitational potential energy. This allows us to rewrite (1) as $m_j\ddot{\mathbf{r}}_j = \nabla_j U = \partial U/\partial \mathbf{r}_j$ for each j, where the gradient is taken with respect to the components of $\mathbf{r}_j$. Then taking the inner product of this expression with $\dot{\mathbf{r}}_j$ and summing over all j, basic vector calculus tells us that

$$\sum_{j=1}^{N} m_j(\dot{\mathbf{r}}_j \cdot \ddot{\mathbf{r}}_j) = \frac{d}{dt}\left(\frac{1}{2}\sum_{j=1}^{N} m_j \mathbf{v}_j^2\right) = \sum_{j=1}^{N} \frac{\partial U}{\partial \mathbf{r}_j} \cdot \dot{\mathbf{r}}_j = \frac{d}{dt}U.$$

Then for some constant with respect to time h which we call the total energy, we have

$$T \equiv \frac{1}{2}\sum_{j=1}^{N} m_j v_j^2 = U + h \tag{5}$$

which simultaneously defines T, the *total kinetic energy.*

Equations (3), (4), and (5) together define 10 first integrals. Assuming all 10 are independent and algebraic, (i.e. at best), they allow us to reduce the number of variables of the N-body system from $6N$ to $6N - 10$. Jacobi appealed to some symmetries to reduce the number to $6N - 12$ by a method called reduction of nodes. But a theorem due to Poincaré [2] stops us at this point:

Theorem 2. *There are no first integrals, algebraic with respect to the time, position, and the velocities only, other than the 10 above.*

This means that the method of solution via use of first integrals fails to solve the N-body problem for $N > 2$; for $N = 3$, at best it may effectively reduce the problem to one in $18 - 12 = 6$ variables. For $N = 2$, as implied by the argument above, the differential equations can be solved explicitly. A change of dependent variables, to $\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$ (the center of mass) and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, the particle separation, makes this easy by standard ODE methods (see [1]). Basically, it is reduced to a 1-body problem, to which we know the solutions.

The above is certainly not a claim that the N-body problem is unsolvable - rather, it states that it cannot be solved for $N > 2$ by a particular method of variable reduction. As we shall see, the problem is amenable to the generalized method of power series solutions. The biggest obstacle is in dealing with the singularities.

2 The 3-body solution

Now we proceed to follow Saari's [5], [4] exploration of Sundman's solution of the 3-body problem. The solution in full-generality is beyond the scope of this paper (and, evidently, only available in its original French). Essentially, our task is to survey the structure of Sundman's proof, offering arguments under simplified conditions in place of rigorous proofs. But as we will see, the tools of basic complex analysis which we have developed play a role in the framework of the general proof.

2.1 Singularities and collisions

Sundman sought a power series expansion solution to the problem. The existence and uniqueness theorem for ordinary differential equations tell us that given an initial condition $\mathbf{r}(t^*) = \mathbf{r}_0$, $\dot{\mathbf{r}} = f(\mathbf{r})$ where f is smooth and $|f(\mathbf{r})| \leq M$ in a neighborhood $|\mathbf{r} - \mathbf{r}_0| < a$, then a unique solution exists for the system for some neighborhood of t^*, namely $|t - t^*| < b$. If we can repeat this at a point near the boundary of $|t - t^*| < b$, we can *analytically-continue* our solution to more values of t. For this problem, we will define a *singularity* to be a time $t = t^*$ where this process of analytic continuation fails. Clearly a *collision*, defined to occur when the system $\mathbf{R}(t) = (\mathbf{r}_1, ..., \mathbf{r}_N)$ approaches a specific finite point $\mathbf{R}^*$ with some pair of points equal, $\mathbf{r}_j^* = \mathbf{r}_k^*$, is a singularity. This is not the only kind of singularity - Loc Hua's term paper (2011) explores one constructed example of such a singularity which ejects a particle to 'infinity in finite time'.

There are two important theorems here:

Theorem 3. *A singularity for the N-body problem occurs at time $t = t^*$ if and only if $r_{min}(t) = \min_{j \neq k} r_{j,k}(t) \to 0$ as $t \to t^*$.*

This follows fairly easily from the existence and unqueness theorem above. Understanding the character of the singularities of the problem is key to finding a general solution that converges for all time. Note that the aforementioned noncollision singularity corresponds to a situation in which the liminf of r_{min} approaches 0 while the limsup is bounded away from 0. But this cannot happen for $N = 3$, which greatly simplifies things. We present the proof of this since it is particularly accessible and gives a taste of the proofs of more complicated theorems cited here:

Theorem 4. *For the 3-body problem, all singularities are collisions.*

Proof. (From [4].)

To prove this, we start by deriving the important *Lagrange-Jacobi equation.* Define $I = \frac{1}{2} \sum_{j=1}^{N} m_j r_j^2$. Differentiating I twice yields

$$\ddot{I} = \sum_{j=1}^{N} m_j \left(\mathbf{v}_j^2 + \mathbf{r}_j \cdot \ddot{\mathbf{r}}_j \right) = 2T + \sum_{j=1}^{N} \mathbf{r}_j \cdot \frac{\partial U}{\partial \mathbf{r}_j}.$$

Now note that the self-potential U is homogeneous of degree -1 in the components of the $\mathbf{r}_j$; so by Euler's theorem, the above simplifies to

$$\ddot{I} = 2T - U = U + 2h = T + h \tag{6}$$

where we have also used our definition of the total energy h. Now we proceed with the proof.

If t^* is a singularity, then we must have $r_{min}(t) \to 0$ as $t \to t^*$. But by the definition of U, we must then have $U \to \infty$ in that limit since one of the denominators gets vanishingly small. But then (6) says that $\ddot{I} \to \infty$ as well, since h is fixed. Then since $\ddot{I} > 0$ near t^*, we must have I approach a limit: $\lim_{t \to t^*} I = D \geq 0$ (I cannot be negative). If $D = 0$ we must have each r_j and thus $\mathbf{r}_j$ approach 0, which is clearly a collision by our definition. Suppose then that $D > 0$.

To ensure this is still a collision, we must first show that $r_{jk} \to 0$ for a fixed pair of particles j, k. For t sufficiently close to t^*, we know that there is some positive E such that the maximum particle spacing is always larger than E: this follows from noting that we can rewrite $I = \frac{1}{2M} \sum_{j<k} m_j m_k r_{jk}^2$ where M is the total mass (easy to check in the $N = 3$ case). But since $r_{min} \to 0$, for t close to t^* we must have the minimum separation always less than $E/2$. If at a time t in this interval we have r_{12} be the minimum, and at another time r_{23} is the minimum, it follows from continuity that there is some time in the interval at which $r_{12} = r_{23} < E/2$; but then we would have $|\mathbf{r}_1 - \mathbf{r}_2| + |\mathbf{r}_2 - \mathbf{r}_3| < |\mathbf{r}_1 - \mathbf{r}_3|$, which violates the triangle inequality. So we conclude that one of the separations r_{jk} does have limit 0.

Since $D > 0$, one of the particles is eventually bounded away from the other 2 - i.e., in a neighborhood of t^*, we can say for instance that $r_{13}, r_{23} > B > 0$ for some constant B. This follows just as above, for if it were not true then two of the separations, say r_{12}, r_{23}, must approach 0; and then for some time we have $r_{12} + r_{23} < r_{13}$, a violation of the triangle inequality. But if particle 3 is bounded away from the others, the equation of motion (1) require that there be a constant A such that $|\ddot{\mathbf{r}}_3| < A$ in a nieghborhood of t^*. Then if we integrate this from any t_1 to any t_2 in that neighborhood, we find that $|\mathbf{v}_3(t_2) - \mathbf{v}_3(t_1)| < A|t_2 - t_1|$; so by the Cauchy definition of convergence, the velocity $\mathbf{v}_3$ must have a finite limit at t^*. Then $\mathbf{r}_3$ is continuous and approaches a limit $\mathbf{L}_3$ as $t \to t^*$. Then since $r_{12} \to 0$, eq. (3) gives us a finite limit for $\mathbf{r}_2$ and $\mathbf{r}_1$ as well.

Thus, all 3-body singularities must be collisions. □

And one more theorem, the Weierstrass-Sundman Theorem, rounds out this line of inquiry: it actually applies to any inverse p force for $1 < p < 3$ (where p denotes one less than the exponent of r_{jk} in the denominator of the force expression in (1)):

Theorem 5. *For the general N body problem, an N-body collision, where* $r_{max} = \max(r_{jk}(t)) \to 0$, *can occur only if the constant angular momentum of the system* $\mathbf{c} = \mathbf{0}$.

So we assume that the angular momentum is nonzero; this means that there are only 2 categories of singularities we must consider in the $N = 3$ case.

2.2 Binary collisions

First we consider real binary collisions, where the system approaches a definite limit in which $\mathbf{r}_j = \mathbf{r}_k$, $j \neq k$. Sundman's treatment of this point is apparently particularly complex. We try to gain an intuition for the steps he took by considering a parallel approach by Levi-Civita, in the simplified central force 1-body problem.

Consider a set of elliptical orbits in the plane, described by eq. (2), where $\epsilon \to 1^-$. This is the limit of a series of elliptical orbits which are being compressed towards a line with the origin (the stationary body) at one end. This is an approximation to a head-on collision with a stationary particle. We expect that the incoming particle will essentially bounce off the other particle in the limit, reversing direction instantaneously.

Now we may use a complex variable argument to gain some insight into how Sundman dealt with these 2-body collisions. We replace $\mathbf{r} = (x, y)$ describing the motion of the incoming particle in the plane with the complex representation $z = x + iy$. Then the equation of motion $\ddot{\mathbf{r}} = -\mathbf{r}/r^3$ becomes (entirely equivalently, since real/imaginary parts agree with the components)

$$\ddot{z} = -\frac{z}{r^3}, \quad r = |z|. \tag{7}$$

We'd like to straighten-out the sudden reversal in the particle's path; that is, in terms of the argument of z, we'd like to find a way to turn a 2π jump into a π jump. The immediately obvious choice is to define a new dependent variable w by $w^2 = z$, or $w = z^{1/2}$ with the cut line directed perpendicular to the incoming motion (so it is not contacted except at 0). So where z reverses direction at 0, the argument of w changes by π: w passes straight through the origin. It seems we have effectively linearized the problem.

To see this more rigorously, we make a change in independent variable as well. Define s by $ds = dt/r(t)$. Note that for some arbitrary twice-differentiable function $f(t)$, by the chain rule we may write

$$\frac{d^2 f}{dt^2} = \frac{r\frac{d^2 f}{ds^2} - \frac{dr}{ds}\frac{df}{ds}}{r^3}.$$

Then applying this to z, we can rewrite eq. (7) as

$$rz'' - r'z' + z = 0$$

where the prime notation indicates differentiation with respect to s. We see that the troublesome factor of r^3 has disappeared; there is no longer an apparent singularity in the equation of motion as $r \to 0$.

We can simplify a bit further. The energy integral corresponding to eq. (5) in this 1-body case is the expression (with appropriate units) $\frac{1}{2}(\frac{d\mathbf{r}}{dt})^2 = \frac{1}{r} + h$. But by the chain rule, $\frac{d\mathbf{r}}{dt} = \frac{1}{r}\frac{d\mathbf{r}}{ds}$. Thus, we have $(z')^2 = 2(r + hr^2)$, which can be written neatly in terms of w as follows. Note that $r = |z| = |w^2| = w\bar{w}$; so $r' = w'\bar{w} + w\bar{w}'$. Together with the fact that $z' = 2ww'$ and $z'' = 2(w')^2 + 2ww''$, the norm of the energy integral simplifies to

$$|2ww'|^2 = 4|w|^2|w'|^2 = 2|w|^2(1 + h|w|^2) \quad \rightarrow \quad 2|w'|^2 = 1 + hr \quad \rightarrow \quad h = \frac{1}{r}(2|w'|^2 - 1).$$

And finally, substituting all of these results into (7) yields

$$w'' - \frac{h}{2}w = 0.$$

In a beautiful turn of events, we have transformed the equation of motion into that for a harmonic oscillator.

As Saari goes on to discuss qualitatively, the same ideas we have used here can be applied using spinors to the problem in 4 variables, which yields a linear solution in $\mathbb{R}^3$. This method regularizes the path through these simple collisions, allowing for analytic continuation.

2.3 Forming the series

The change of dependent variables from t to s, where $ds = dt/r(t)$, is the key to Sundman's local series solution and its avoidance of the binary collision issue. We can gain some understanding of why it works here by considering it in the context of the central force problem again.

Suppose two particles collide in one dimension. We call x the distance between the particles; then the equation of motion is $\ddot{x} = -1/x^2$. If $\dot{x}$ is initially negative, this equation of motion makes it clear that this will remain so. We can get the energy integral for the problem by multiplying by $\dot{x}$ and integrating: $\dot{x}^2 = \frac{2}{x} + 2h$. By theorem 3, we know that if the collision occurs at t^* the variable x vanishes in the limit; so near t^*, we have $x(\dot{x})^2 = 2 + 2hx \sim 2$. Then $x^{1/2}\dot{x} \sim \sqrt{2}$; or, integrating this, $x(t) \sim A(t^* - t)^{2/3}$ for a constant A.

It can be shown that all binary collisions are related to what are known as algebraic branch points, a concept in multi-valued function theory. This statement is encapsulated in the solution to our simplified problem: in a neighborhood of t^*, the separation x is described by the series

$$x(t) = \sum_k a_k (t^* - t)^{2k/3}.$$

We can see this by defining a dependent variable X by $X(t)(t^* - t)^{2/3} = x(t)$. Let $t^* = 0$. By some simple differentiation and substitution, we find that the equation of motion can be written for X in the form

$$t^2\ddot{X} + \frac{4}{3}t\dot{X} - \frac{2}{9}X = -X^{-2}.$$

To convert this to a form we may solve, let $s = t^{1/3}$. Then using the chain rule, and denoting derivatives with respect to s by slashes, we easily convert this to the equation

$$s^2X'' + 2sX' - 2X = -9X^{-2}.$$

Then the method of Frobenius (generalized power series solution) yields a series solution of the form

$$x(t) = \sum_k a_k t^{2k/3}$$

as claimed. Sundman's series in $N = 3$ is also of this form, in powers of $t^{2/3}$.

2.4 Complex singularities and the global solution

So Sundman has a local solution to the 3-body problem, obtained by restricting $\mathbf{c} \neq \mathbf{0}$ and by performing variable substitution similar to that above to regularize what would otherwise be singularities. But as we know, the radius of convergence of a power series of a real variable depends on the location of singularities in the complex plane around the expansion point. It is easy to see that there are imaginary singularities (collisions at complex-valued times) in the N-body problem generally. Consider again the central force problem; specifically, the elliptical solution described by $r(u) = a(1 - \epsilon\cos(u))$ and $t = u - \epsilon\sin(u)$. Now substitute for t the complex variable $t_1 + it_2$, and similarly let $u = u_1 + iu_2$. Then consider the expanded form of $\cos(u)$:

$$\cos(u) = \frac{e^{iu} + e^{-iu}}{2} = \frac{e^{iu_1}e^{-u_2} + e^{-iu_1}e^{u_2}}{2}$$

So if we set $u_k = 2k\pi + i\cosh^{-1}(1/\epsilon)$, the above becomes

$$\cos(u_k) = \frac{e^{-\cosh^{-1}(1/\epsilon)} + e^{\cosh^{-1}(1/\epsilon)}}{2} = \cosh(\cosh^{-1}(1/\epsilon)) = \frac{1}{\epsilon}$$

which yields $r(u_k) = 0$. So clearly collisions can occur at imaginary t; in this case, $t_k = u_k - \epsilon \sin(u_k)$. And there is no guarantee that these are well-behaved enough to be 'regularized-out' as with real binary collisions.

But Sundman found a way around this complication. As we hinted at in the proof of theorem 4, one can put some lower bounds on the quantity $r_{max}(t) = \max(r_{jk}(t))$. Sundman proved that for each orbit with $\mathbf{c} \neq \mathbf{0}$, there is a constant $D > 0$ such that for all t we have $r_{max}(t) \geq D > 0$. A difficult argument proves that there is a strip $S = \{s : |Im(s)| \leq \beta\}$ on which the system has no singularities. Then we can define a conformal mapping from s to a new variable, σ, by

$$\sigma = \frac{e^{\pi s/2\beta} - 1}{e^{\pi s/2\beta} + 1}$$

. We claim this is a conformal map from the strip S to the unit disc. We know that e^z conformally maps the strip $|Im(z)| < \pi/2$ to the right half plane, and the linear fractional transformation $(1 + z)/(1 - z)$ maps the unit disc to the right half-plane. Inverting the latter tells us that the map $g(z) = (z - 1)/(z + 1)$ maps the right half plane to the unit disc. And $h(z) = e^{(\pi/2)z/\beta}$ must map S to the right half plane. Thus the conformal composition $g(h(z)) = \sigma(z)$ must map S to the unit disc.

By construction, there are no singularities in the unit disc of σ. Thus, the power series in the new variable is convergent there. So it follows that since the disc is conformally equivalent to the strip, we have a power series convergent for all time.

There are large caveats. It turns out (perhaps not surprisingly) that the series Sundman derived converges incredibly slowly. This is largely a result of the changes in variable. The first, from t to $(t^* - t)^{1/3}$, effectively slows the convergence (an interval of some length in t is mapped to a longer one in s generally). And the final change of variables via the conformal map to the unit disc results in many points being mapped near the boundary of a region of convergence (which generally means slower convergence).

3 Discussion

Sundman's solution is an entirely valid solution to the 3-body problem as posed above. And, if one ignores higher-order collisions, it is extended fairly readily to solve the general N-body problem [3]. Yet besides building upon the analytical framework used today to examine the problem, this solution has essentially failed to improve our understanding of the problem. Its abysmally slow convergence has rendered it practically useless. As Diacu [2] points out, this can be viewed as an interesting cautionary tale for mathematics. In this case, the question originally posed evidently did not demand enough of a solution. Despite the fact that power series solutions to systems of ODEs such as this are often of great utility, the general consensus is that this particular problem is not susceptible to this method.

But it is important to note that Sundman's result is part of an ongoing attack which, while not yet having yielded a satisfying analytic solution or answer to the biggest questions, has inspired the creation of a strikingly large amount of mathematics. In particular, attempts to solve this problem are related historically and motivationally to Cauchy's results in complex analysis, to the argument principle, and to Rouche's theorem ([5], pg. 118).

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