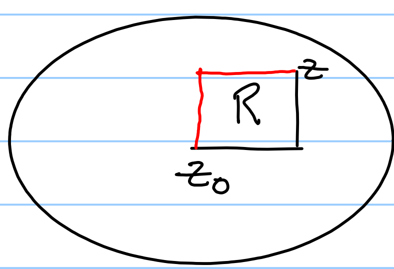
Cauchy Integral Theorem, Simple version

Suppose W is an open connected set with a point $z_0 \in W$ so that for every $z \in W$, the rectangle with opposite corners z and z_0 and sides parallel to the axes belongs to W. Then if f is complex analytic in W

$$\int_C f(z)\,dz = 0 \quad \text{for all closed curves } C \text{ in } W.$$

Pf: We will show that $f(z)$ has an antiderivative. $F'(z) = f(z)$.

Let $z \in W$ and let $\int_{z_0}^{z} f(\zeta)\,d\zeta = F(z)$ be the integral from z_0 to z along a pair of sides of the rectangle with opposite corners z and z_0.

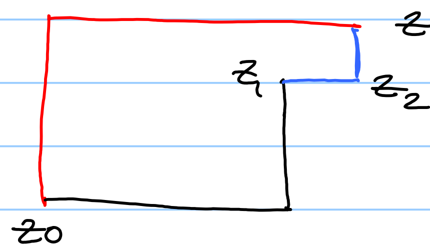


(Use either the red or black sides.)

$F(z)$ is well-defined since by Goursat's theorem

$\int_{\partial R} f(\zeta)\,d\zeta = 0$. Now we prove $F'(z) = f(z)$.

Fix $z = z_1$. Compare $\dfrac{F(z_1+\Delta z) - F(z_1)}{\Delta z}$ to $f(z_1)$.



blue edges $= [z_1, z_2] \cup [z_2, z_1+\Delta z]$

$= B$

$$F(z_1+\Delta z) - F(z_1) = \int_{\text{blue edges}} F(\zeta)\,d\zeta = \int_B F(\zeta)\,d\zeta$$

$$\int_B f(z_1)\,d\zeta = [z_1+\Delta z - z_2 + z_2 - z_1]\, f(z_1)$$

$$= f(z_1)\,\Delta z$$

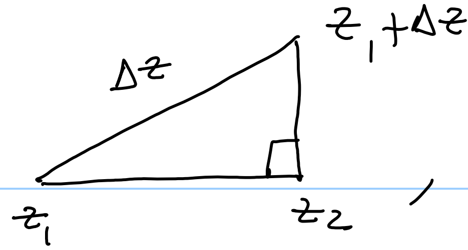
So, $\dfrac{F(z_1+\Delta z) - F(z_1)}{\Delta z} - f(z_1)$

$$= \left[\int_B [f(\zeta) - f(z_1)]\,d\zeta\right] \Big/ \Delta z$$

$|f(\zeta) - f(z_1)| < \epsilon$ if $|\Delta z| < \delta$.

Hence

$$\left| \frac{\int_B [f(\zeta) - f(z_1)]\,d\zeta}{\Delta z} \right| \le \frac{\epsilon}{|\Delta z|} \Big(|z_2 - z_1| + |z_1+\Delta z - z_2| \Big)$$



$$|\Delta z|^2 = |z_2 - z_1|^2 + |z_1 + \Delta z - z_2|^2$$

$$\therefore \quad |z_2 - z_1| \le |\Delta z|, \quad |z_1 + \Delta z - z_2| \le |\Delta z|$$

$$|z_2 - z_1| + |z_1 + \Delta z - z_2| \le 2|\Delta z|$$

So

$$\left| \int_B \frac{[f(s) - f(z_1)]}{\Delta z}\, ds \right| \le 2\epsilon.$$

Thus $F'(z_1) = f(z_1)$.

Now if C goes from p to q,

$$\int_C f(s)\,ds = F(q) - F(p)$$ and if $p = q$

i.e. if C is a closed curve,

$$\int_C f(s)\,ds = 0. \qquad \text{QED}$$