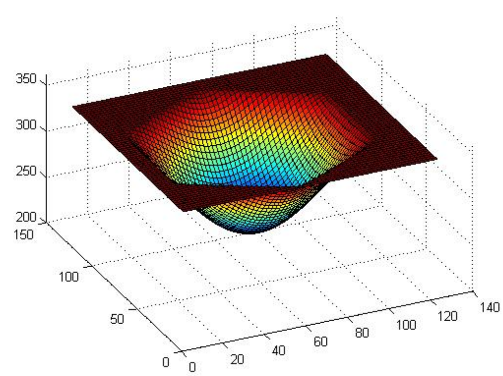
In this paper, we present computerized models to simulate efficient methods for fitting brownie pans, also called "packing", into a fixed-width oven. We also apply analytic and numerical methods to solve the heat equation. We first consider regular polygonal pans to simplify the problem and then gain intuition on more complex shapes. The two parts of the model we focus on, packing and heat distribution, are then weighted together as a function of n, the number of sides of the pan, and maximized. We then draw conclusions about which shape of pan is ideal in order to maximize this weighted function.

We initially take a naive approach to optimizing the pan configuration in a fixed rectangular oven. We inscribe the pans into minimal bounding rectangles, which are then tiled. This approximation gives a lower bound to how many pans will fit. We then refine the model to one in which we inscribe specific parallelograms into the pan, which are then tiled, this results in a more dense packing.

We then model the heat distribution throughout the pan as a two-dimensional differential equation with boundary conditions simulating room temperature and oven baking. Analytic solutions to the square and rectangular pan are provided, and an explicit finite difference scheme is used to model heat diffusion across arbitrary polygonal pans by projecting a lattice of points over the polygon. Results of this computation are presented, and a metric is established to measure how uniformly a pan heats while in the oven. This metric, combined with the results from our packing computation, are used to give output for the weight equation presented in the problem statement.



Determining the Optimal Pan for Baking Brownies

Control #21624

February 4, 2013

Abstract

In this paper, we present computerized models to simulate efficient methods for fitting brownie pans, also called "packing", into a fixed-width oven. We also apply analytic and numerical methods to solve the heat equation. Specifically, we use the Explicit Finite Difference Method to give a numerical approximation for how polygonally shaped brownie pans experience temperature increase at different locations. After developing each of these models, we discuss the trade-offs between them, and provide a reasonable design for a brownie pan that takes into account each models' results.

Contents

1 Introduction.

In this paper, we develop computerized models for efficiently arranging brownie pans in an oven, also called "packing", and for determining how uniformly a brownie pan and its contents will heat up in an oven.

1.1 Outline of Approach.

The first topic presented in this paper details our exploration into brownie pan packing algorithms. It consists of a precise formulation of exactly what is meant by a "good" packing algorithm. Once this is defined, we state two algorithms developed, and discuss their results for various different pan shapes.

Following this, we give a similar discussion on what it means for a brownie pan to have a "good thermal uniformity", and explain our models for finding this quantity given a brownie pan.

Finally, we discuss the trade-offs inherent in these two quantities. Informally speaking, we will see that shapes with fewer sides, such as squares and hexagons, lead to increased space-efficiency but shapes with more sides, such as the circle, lead to increased thermal uniformity.

1.2 Assumptions.

Modeling the packing and heating of brownie pans in full generality is a wildly impractical task, so as a result, we are required to make reasonable assumptions in order to simplify many of the calculations made. These assumptions are enumerated below. We later address ways to relax some of these simplifications.

- **We deal with a fixed oven size and pan area.** This is what the reader might expect. Our models are robust enough to handle variable area pans and variable width and length ovens, however, in reality, ovens and pan sizes do not change, so for every simulation, we fix a pan and oven size, and work with these dimensions.

- **Regular n-gon shaped pans will yield an appropriate solution.** It is possible to imagine that there are some very irregular shapes that behave reasonably as brownie-cooking pans. We believe this case to be unlikely given the current state of bakeware. Additionally, it is unlikely that consumers will be prompted to buy cookware that is "unconventional", so we restrict our models to regular polygons.

- **A 2-dimensional analog is an appropriate representation.** In order to make our model, we had to adapt to the time and computational restraints imposed upon us. Rather than finding solutions to the 3-dimensional heat equation, a much more involved procedure, we elected to instead solve the 2-dimensional heat equation, and represent our pans in two dimensions.

2 Finding an Optimal Pan Configuration.

2.1 Defining a Metric.

We first need some notion of what it means to have a "good" pan configuration. The approach we will take is that a better pan configuration has less empty space. The best way to achieve this is by rearranging and being able to fit a whole new pan into the defined space. Barring this, we will define a packing to be better if it is local more compact, that is there is less empty space near the pans. This can be quantified by looking at the smallest polygon of smallest perimeter that covers the set of the polygons, and taking a ratio of the area of the polygons and the covering.

$$\frac{A_n}{A_c},$$

where A_n is the area of the n-gons and A_c is the area of the covering. A perfect tiling of tessellated squares has ratio 1, which is as good as the ratio can be. The blue squares below have a ratio of 0.857, while the magenta squares have a ratio of 0.8, so the areas of the polygons are less than the coverings. The magenta configuration has more empty space locally, and therefore is defined to be worse.

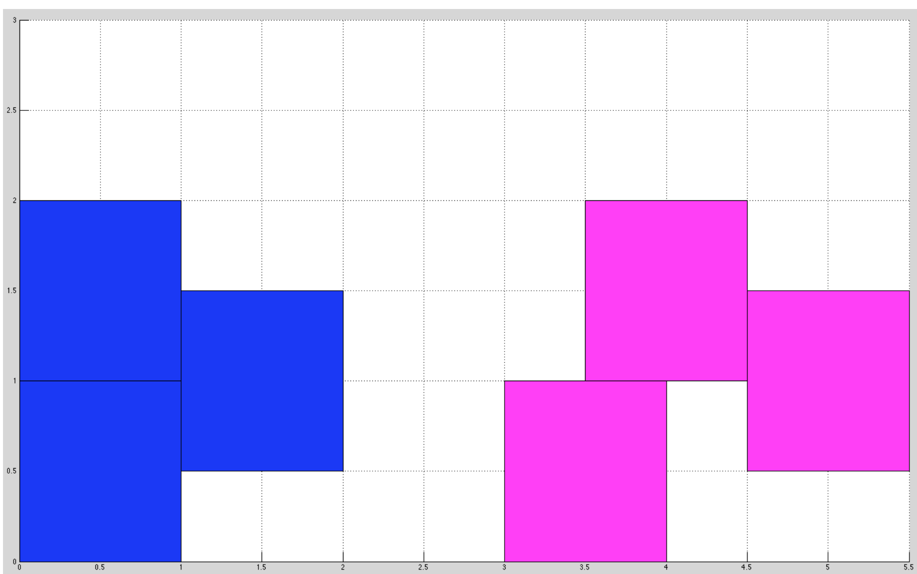


Figure 1: The group of blue squares on the left is more locally compact, while the magenta squares on the right are less. So we define the configuration on the left to be better.

2.2 Naive Method.

We first want to take an intuitive approach. Rectangles tile very easily within any other given rectangle of larger size. This method involves finding the minimum bounding rectangle (MBR) for any n-gon. Then, tiling this bounding rectangle will give us an approximation for how many n-gons we can fit into the same space. This method is inefficient because the bounding rectangle will always have some empty space because the n-gons are all convex. However, it will guarantee that we can fit at least as many n-gons as we can fit rectangles.

2.2.1 Constructing the Rectangle.

In order to find the MBR we first are given a regular n-gon of area A. It is known that the MBR will have at least one side that is collinear to a side of the n-gon. So the first step is to choose a side of the given n-gon to use as a starting line for the box. Then we just construct two more side, each 90° from the starting line, such that the new side intersect the n-gon at either only one point or is collinear to another side, since the n-gon is convex this criteria is going to allow us to have the smallest side that still bounds the shape. The last step is to just close off the rectangle with a final side, which is once again, either intersecting at a single point or collinear to another side. This process is repeated, using each of the n sides as a starting side, then the construction with the smallest area is chosen as the MBR. This can be made more efficient by noticing that when n is divisible by four, the n-gon is symmetric such that it does not matter which side is used as the starting point, the constructed rectangle will be the same size.

2.2.2 Tiling the Rectangles.

Now that we have this rectangle, we can reduce the problem to simply tiling the given space with rectangles, rather than n-gons. This is done by lining the first rectangle up with the corner of the space, and then tiling over until no more will fit. This is repeated in rows, until a new row itself could not fit. At this point we are done and just need to count how many rectangles there are, this is a lower bound for how many n-gons will fit into the space.

2.3 Refinement of Naive Method.

While the method of MBR is a simple algorithm, it is certain not the most efficient at packing. There is wasted space when fitting a rectangle. In order to have a better packing algorithm, we turn to a paper written by David Mount, *The Densest Double-Lattice Packing of a Convex Polygon* [1]. This double-lattice method begins by creating a parallelogram inside of the n-gon. The parallelogram is special though, in that it has sides equal to the circumradius of the n-gon if n is even, or side equal to half the sum of the circumradius and the apothem if n is odd. This way, it is guaranteed that in a tiling of these parallelograms, every n-gon will only overlap the adjacent parallelogram up to

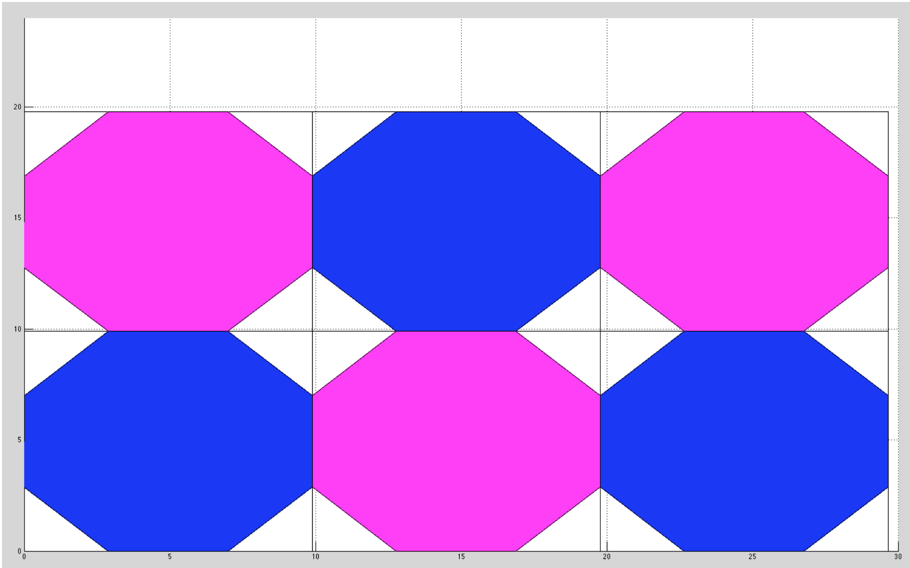


Figure 2: An example of the naive algorithm for $n = 8$ with a fixed area of 81 square inches. The 6 pans are packed into a 30 by 24 inch oven.

halfway. So if we position one n-gon imposed over every other parallelogram, in a checkerboard pattern, we can achieve a very dense packing of n-gons.

With this dense packing algorithm, we are often able to see an improvement of an extra pan over the MBR method. By inscribing a parallelogram inside the shape as opposed to creating a bounding rectangle, we can minimize the amount of empty space between pans.

3 Analyzing Thermal Uniformity.

For those truly invested in premium brownie cooking dynamics, thermal uniformity cannot be overlooked. The fundamental difference between the different types of pans stems from the basic geometry each has. The square pans burn more because the edges of the pan stay noticeably hotter than the rest of the pan for a period of time during the cook. For a normal spread of brownie mix, this uneven heat distribution from the pan leads to an uneven heating of the brownies.

3.1 Analytic Solutions for Square and Circular Shaped Pans.

Our first approach to the pan modeling problem is to examine the first two basic types of pans: The circle and the square. We begin by estimating the pan is

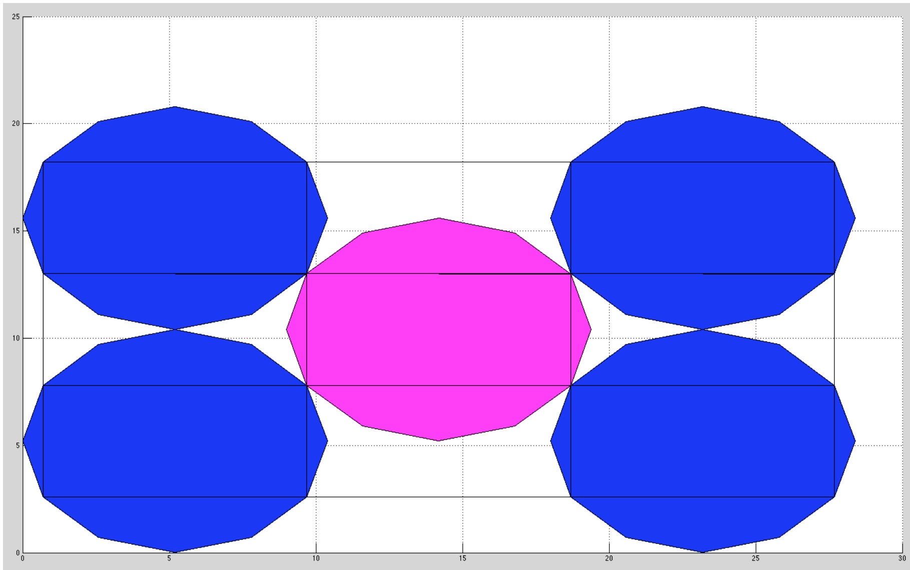


Figure 3: An example of the refined algorithm for $n = 12$ with a fixed area of 81 square inches. The 5 pans are packed into a 30 by 24 inch oven.

made of a material similar to iron or stainless steel, and as such has a thermal diffusivity constant of $\alpha = 4 * 10^{-6}$. We begin modeling the square pan as a two dimensional square with the heat equation $\partial_t u(x, y, t) = \alpha \partial_x^2 u(x, y, t)$ and boundary conditions consistent with the temperature of an oven. This amounts to a Boundary Value Problem for temperature $u(x, y, t)$ over a square of length a and height b with the constraints of the temperature of the oven

$$\partial_t u(x, y, t) = \alpha \partial_x^2 u(x, y, t), u(0, y, t) = u(x, 0, t) = u(a, y, t) = u(x, b, t) = 350$$

and the initial condition, which represents room temperature,

$$u(x, y, 0) = 85.$$

The analytic solution[3] of this system is of the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{m,n} \sin(\frac{m * \pi}{a}) \sin(\frac{n * \pi}{b}) e^{-\lambda_{m,n}^2 t}$$

where

$$A_{m,n} = \frac{4}{ab} \int_0^a \int_0^b u(x, y, 0) \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) dy dx$$

and

$$\lambda_{m,n} = \sqrt{\frac{m\pi}{a} + \frac{n\pi}{b}}$$

For square pans, the temperature distribution behaves traditionally as we would expect, the following diagrams show the temperature distribution of a square pan as it heats up inside the oven (See table 1)

Notice that even though the shape remains relatively the same at later times the axis dictating the temperature of our square are rapidly approaching that of the oven. However, the rate of convergence is what concerns us. Even though the pan cooks brownies perfectly well, we are seeking a pan that heats up more uniformly than this.

The circular pan can also be solved for analytically, using a similar method of solution to the partial differential equation. Again, we have the heat equation $\partial_t u(x, y, t) = \alpha \partial_x^2 u(x, y, t)$ applied to a circular region with the initial conditions $u(x, y, 0) = 85$ and the boundary conditions, best given in polar coordinates $u(r, \theta, t) = 350$. As the boundary condition may indicate, it is best to solve this problem in polar coordinates. We can take advantage of the additional cylindrical symmetry of both our partial differential equation and the region and express the solution[2] in terms of Bessel functions:

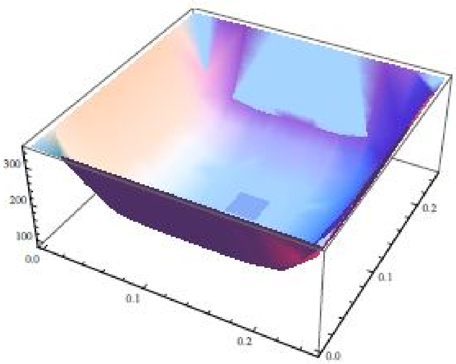
$$U(r,t) = (290)(1 - 2\sum_{n=1}^{\infty} \frac{J_0(\frac{\alpha_n r}{a})}{\alpha_n J_1(\alpha_n)} e^{-\alpha_n^2 kt/a^2})$$

where J_0 is the Bessel function of the first kind, and α_n is the n^{th} root of the Bessel function. Although notationally confusing, here our α, or thermal diffusivity constant, is the k in the formula above. We should also note that in the solution we have scaled the room temperature to be zero degrees and the temperature of the oven to be 290. The following plots (see table 2) are assuming the pan is about 10 centimeters in radius, using these conditions.
Again, the last plot looks similar to the previous ones, but the axis are scaled to less than one in one thousand degrees. By twenty minutes the temperature is, for all intents and purposes, uniform. However, we should note the difference between the circular pan and the square pan at the two minute mark. Both pans are far closer to the temperature of the oven, but the circular pan has a smaller difference between the center of the pan and the edge than the square pan. This is the fundamental difference between these two extreme pan types.

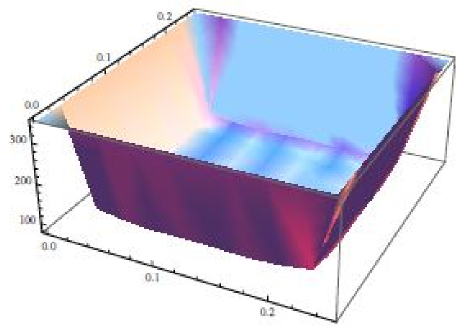
3.2 Solutions for More Shapes Using Finite Difference Analysis.

After calculating the analytic solution to the partial differential equation for both the square and the disk, a significant problem began to emerge. Despite the high symmetry and easily expressed boundary conditions, the solutions were still fairly difficult to produce. This problem becomes incredibly difficult if the shape we are considering measuring temperature over shifts to some other polygon with n sides. Instead of going on a case-by-case basis for solving each polygon's analytic form, we decided to shift to a popular method for solving partial differential equations: numerical methods.

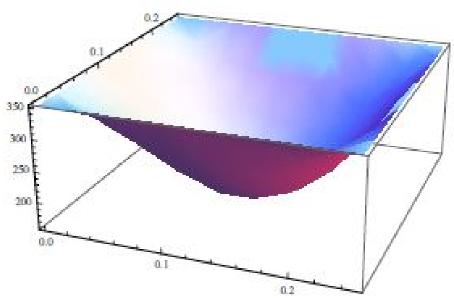
Table 1: Temperature distributions for a square pan in an oven



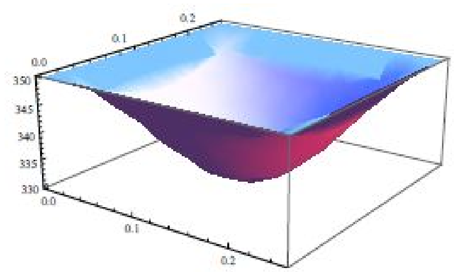
Temperature at 10 seconds



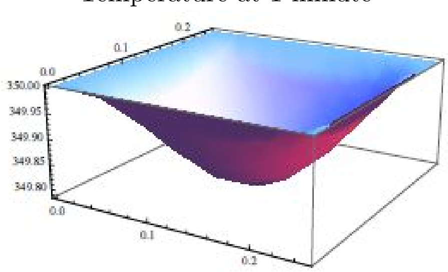
Temperature at 25 seconds



Temperature at 1 minute

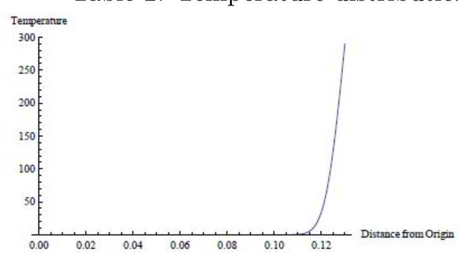


Temperature at 2 minutes

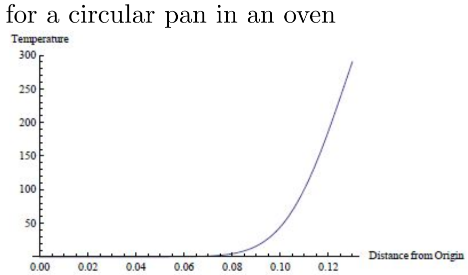


Temperature at 10 minutes

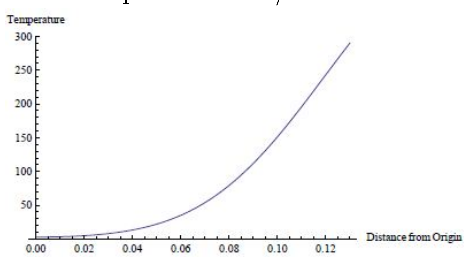
Table 2: Temperature distributions for a circular pan in an oven



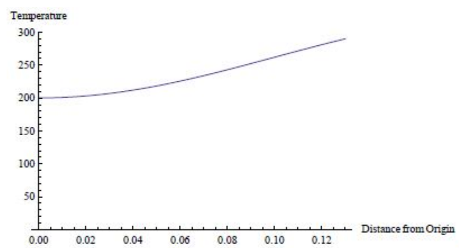
Temperature at 1/2 seconds



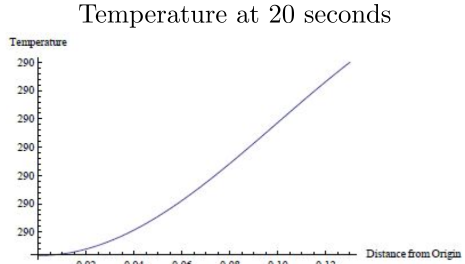
Temperature at 5 seconds



Temperature at 20 seconds

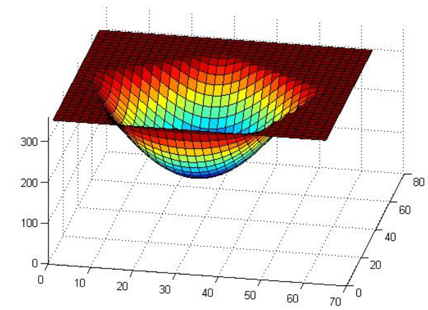


Temperature at 2 minutes

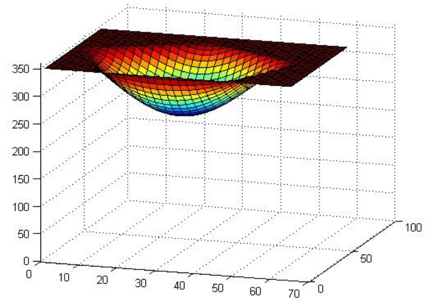


Temperature at 20 minutes

Table 3: Temperature distributions over a pentagon



pentagon at 5 seconds



pentagon at 10 minutes

The Explicit finite difference method involves discretizing the differential equation into tiny, but finite, steps through space and time. The spatial and temporal differences can be computed through the Taylor expansion of our temperature function:

$$\alpha\nabla^2 U(x,y,t) = \partial_x^2 U + \partial_y^2 U = \alpha\left(\frac{U_{m+1,n} - 2U_{m,n} + U_{m-1,n}}{\delta x^2} + \frac{U_{m,n-1} + U_{m,n+1} - 2U_{m,n}}{\delta y^2}\right)$$

and

$$\partial_t U = \frac{U^{p+1} - U^p}{\delta t}$$

where the p superscript denotes the pth time step, much like m and n represent the x and y spatial coordinates. Together, and with some arithmetic manipulation and equal mesh size, this becomes the forward finite difference scheme:

$$U_{m,n}^{p+1} = \frac{\alpha\delta t}{\delta x^2}(U_{m+1,n}^p + U_{m-1,n}^p + U_{m,n+1}^p + U_{m,n-1}^p) + (1 - \frac{4\alpha\delta t}{\delta x^2} U_{m,n}^p)$$

This method gives us the flexibility of evaluating arbitrarily shaped pans, but we restrict our attention to realistically shaped regular polygons with n sides in this paper. This algorithm, when applied to a lattice of points superimposed on a polygonal image, can simulate heating a pan shaped as that polygon. Each point lying outside the polygon is held at 350 degrees, while points inside the polygon are initially at 85 degrees. Then, this algorithm is applied to all of the points inside the polygon, simulating the heat distribution across quantized pieces of the polygon. The temperature of a pentagon at two temperatures is included in table 3.

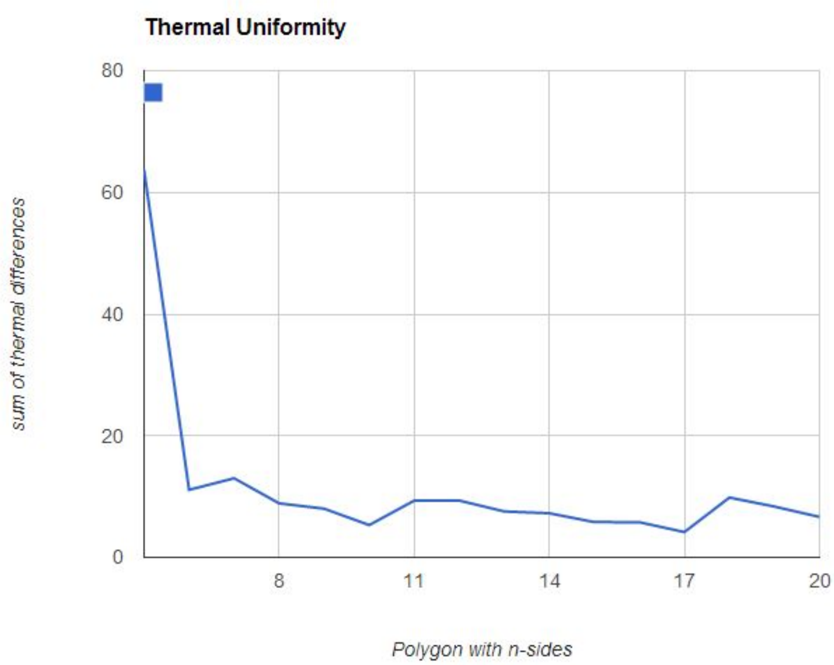
4 Optimization Trade-offs.

4.1 Defining a Metric for Thermal Uniformity

To establish a good idea for weighting the advantages and disadvanages of packing and thermal uniformity we first need a metric. For thermal uniformity, a good way to go about measuring this is to measure exactly how far off the temperatures in the pan are from each other. However, all of the pans heat close to uniformly by about 10 minutes, so the important interval of temperature difference is before then. To cook brownies equally, we want to minimize the difference in heat between the center and near the edge of the polygonal pan for that interval. To this end, we can create an approximate integral of the difference in heat distribution between the two points on the pans by summing over discrete time steps from first putting the pan in the oven to around ten minutes. This formula will take the form of

$$\sum_{t=0,5,10\ldots}^{600} U_{center} - U_{midway}$$

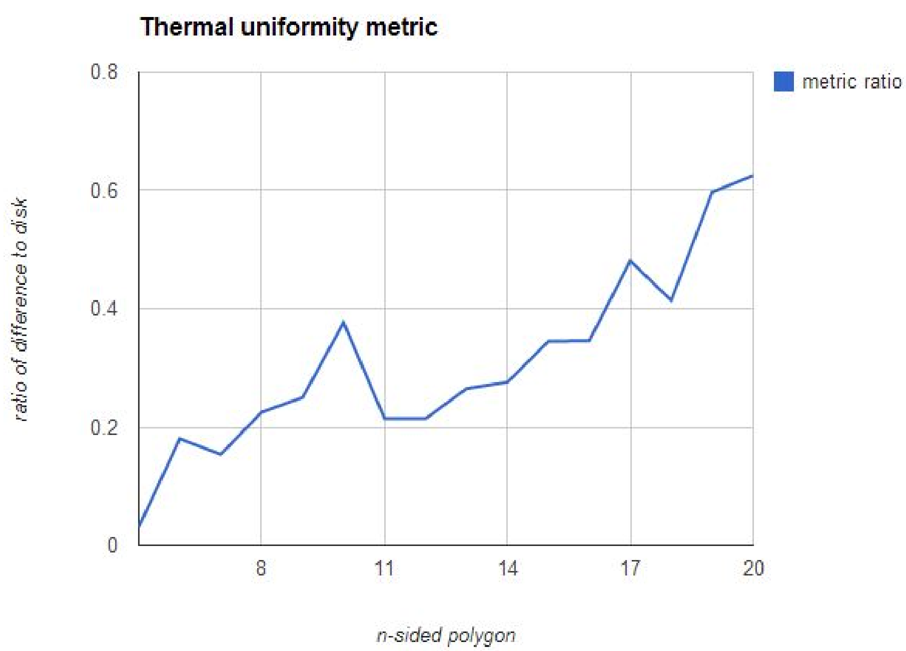
where U_{midway} is the temperature half way in between the edge of the pan and the center. Directly using the edge of the pan will lead to a constant value for every pan because the boundary conditions for each pan are constant. The results for the first twenty polygons, starting at a pentagon are:



To develop a good metric with this simulated data we need to understand what an optimal pan should be. It would be great if the pan heated uniformly instantly, but that is unrealistic. The best type of pan we have is going to be a disk, so a proper ratio between the thermal difference of the disk (using the same sum as above) and the thermal differnece of the polygonal pan will give us a number, between 0 and 1, that tells us how close to the thermal uniformity of the disk our polygonal pans are! The formula for this is,

$$\frac{H_n}{H_d} = \frac{\sum_{t=0,5,10...}^{600} U_{n-gon}^{center} - U_{n-gon}^{midway}}{\sum_{t=0,5,10...}^{600} U_{disk}^{center} - U_{disk}^{midway}}$$

which yields the following ratios:



Notice that as the polygonal shapes become more convex, their ratio approaches one, meaning that the thermal uniformity of the polygons also approaches the thermal uniformity of the disk. This corroborates our intuition about the relationship between the geometrical similarity between the increasing sides of the polygons and the disk.

4.2 Optimization

Now that we have our hands on two metrics, one for packing and one for thermal uniformity, we can create a weighted function of n. Using a weight factor p where

$p < 1$ we get the following equation,

$$v = p\left(\frac{A_n}{L \cdot W}\right) + (1-p)\left(\frac{H_n}{H_d}\right),$$

where A_n is the area of the packed n-gons, $L \cdot W$ is the area of the oven, and H_n/H_d is the thermal uniformity metric. Our goal now is find which integer n maximizes v for any given weight.

Taking $p = 1/2$, that is an even weight on thermal uniformity and on packing, in a standard size oven, with fixed area equal to a 9" by 9" pan, we get the result that $n = 8$ will maximize v. This makes sense because the octagon is the shape with the most sides that will still be able to fit six times into the oven. Thermal efficiency rises as n$\to\infty$, while packing is worst as n$\to\infty$, so if we bias the weights to be all towards packing, $p = 1$, then we see that in the standard case, $n = 4, 6, 8$ will maximize v. If we then add a slight bias to heat, that is increase p by a small amount, we can break the tie by notice that $n = 8$ has a much better thermal uniformity. On the other hand, if we set $p = 0$, that is weighting thermal uniformity over all else, we get that a circle, or $n \to \infty$, will maximize v.

As $L \cdot W$ is varied we noticed that the packing efficiency did not change very much. That is, unless we went to the extreme and had an oven which was very skinny in one direction while being long in the other, then it could be the case that only squares could fit into the oven, while any larger shape would have too big a circumradius to even fit one. In our model the area of the oven does not change the thermal efficiency at all. So our model generalizes to sufficiently nice dimensions of oven, certainly all commercial ovens.

5 Improving the Model.

The circumstances under which our models were developed required the imposition of several limitations that are not ideal. We discuss these in detail below, separating improvements based on how they could affect the thermal model, and how they could affect the pan packing model.

5.1 Generalizing and Expanding the Pan Packing Model.

5.1.1 Finding more Accurate Algorithms.

Due to time constraints, the algorithms used for packing needed to be simple enough to implement in a time-efficient manner, as well as robust enough to provide accurate results. We feel as though this line was well-walked, but recognize that a genetic algorithm may have been well suited to this problem. Genetic algorithms allow for efficient arrangements of inputs to be found, using previously attempted arrangements. The results of such a procedure could lead to a packing of pans that does better than our algorithms have provided. However, genetic algorithms are very difficult to implement correctly, and require more pre-requisite knowledge than was available. So had there been more time and resources, this may have been an avenue worth exploring.

5.1.2 Relaxing the Regularity Assumptions on Pan Shape.

We began by assuming that working with regular polygon shaped pans would yield appropriate results. While this assumption is ostensibly correct, there are some compelling arguments that suggest a better tiling may be achieved if irregular polygons are used [1]. Moreover, had their been time to investigate this further, the refined algorithm for packing described above could have been modified with relative ease.

5.2 Providing More Detail in the Model for Heat Transfer in the Oven.

5.2.1 Generalizing the Model to 3-dimensional objects.

Perhaps the largest shortcoming of our thermal model for the process of brownie baking is our use of the 2-dimensional heat equation. The boundary conditions of a 3-dimensional model would more accurately reflect how heat flows through the pan. Such a method could be implemented by defining a bounding cube around the pan, instead of a rectangle, and then making a 3-dimensional lattice from this bound, as opposed to the 2-dimensional one we used. Our choice to forego this additional level of accuracy was colored by its greater complexity and thus required time commitment. Given more time, this method of implementation would result in a more accurate model of the pan during its cook time.

5.2.2 Accounting for the Effects the Brownies.

Heat travelling through the brownie inside the pan means that heat can leave and transfer through the pan. This would result in slightly different dynamics of heat flow, it would be advantageous to incorporate these into the model. Performing the additional calculations required to model this would have been beyond the scope of what was reasonable to implement within the time constraint defined by the competition.

5.2.3 Develop more Accurate Boundary Conditions.

In the current model, the boundary conditions are assumed to be a constant temperature, which is exactly the temperature to which the oven was preheated. While this serves as a reasonable approximation, it would also be beneficial to incorporate a less static environment. For example, many ovens have convection currents that distribute air throughout the oven in currents, and our model does not account for such an event. As with the rest of these suggestions for improvement, the assumption of static ambient air within the oven was made to simplify the model in order to obtain results in a consistent and timely manner.

6 Conclusion.

Our model revealed several things about the dynamics of baking brownies and the problem posed by changing the pan shapes. Heat flow, although given in a easily expressible differential equation, can pose a serious computational hurdle when explicit results are needed. Analytic solutions to arbitrary shapes are unrealistic to obtain quickly, leading to our discovery of fairly accurate numerical methods of computation. With regard to packing efficiency, our models determined that optimal packing could be achieved with pans shaped as squares, hexagons, or octagons. Odd-numbered shapes are more difficult to compute packing arrangements for than even numbered ones. Despite this, we did come up with a viable relationship between polygonal shapes and thermal uniformity so that somebody with a preference for a particular shape or a desire for well-cooked brownies can use our math to help decide what pan shape to get.

7 References

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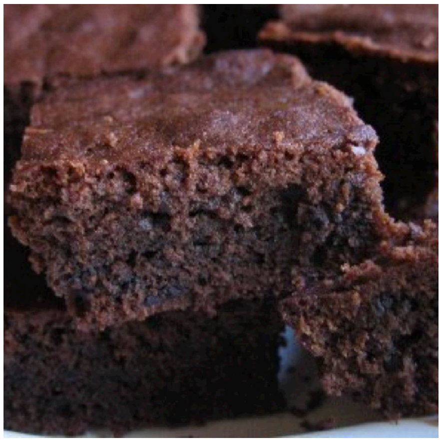
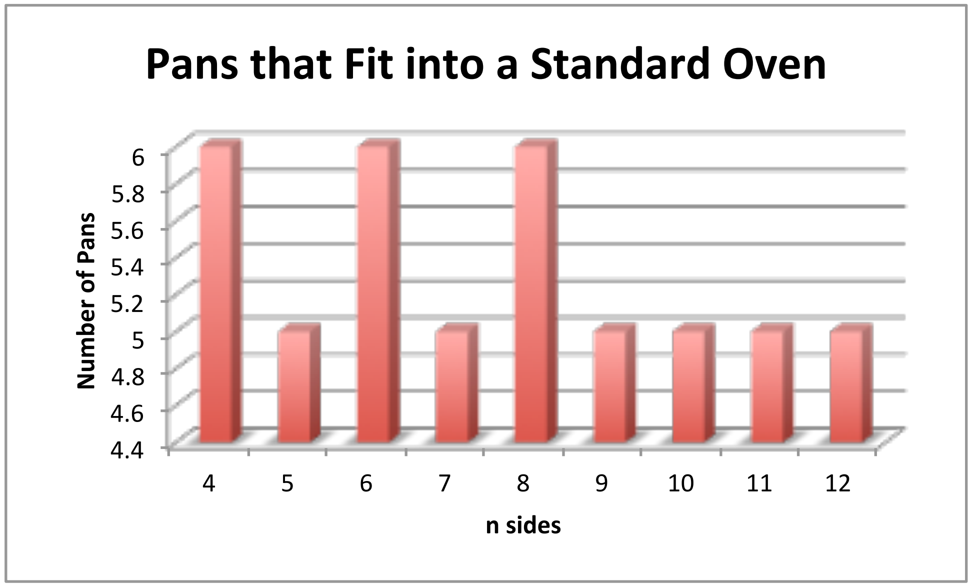
BROWNIE GOURMET MAGAZINE





Do you enjoy eating evenly baked brownies? Well, the time is now. Emerging research in heat distribution shows that not only can we have brownies that aren't overcooked on the edges, but that are space efficient in your oven!

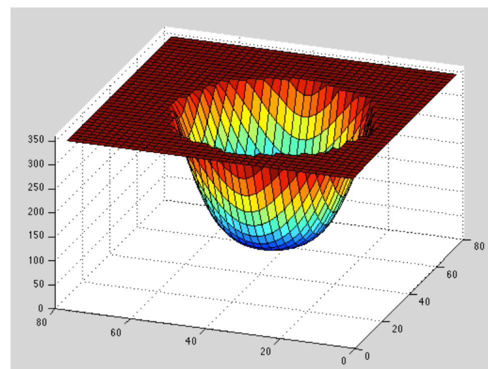
The new octagonal brownie pans from *Brownie Pans Unlimited LLC* is paving the way on the cutting edge of pan technology. Our pans will optimize a carful balance between heat distribution and packing efficiency.
Our pans have a fixed area equal to that of a 9" x 9" baking pan, so you won't have to alter the amount of ingredients you purchase

- 8-sided regular polygon.
- Stainless Steel for a mid-range heat diffusion, for staying efficient while maintaining integrity.
- Quick diffusion of heat from corners of the pan (bottom right).

DESIGNED FOR EFFICIENCY:

As you can see in the chart above, the more sides you add to the pan, the harder it is to fit into a rectangular oven. Conversely, as you add more sides, you even out the heat distribution, thus providing you with less overcooked brownies. Our pan is specially designed to maximize the balance between these factors.

The shape helps to alleviate pockets of heat in the corners, where as a square pan will heat up faster in the corners and tend to overcook your food. With our octagon pan you can be at ease knowing



that the heat is being uniformly distributed away from building up in the corners, resulting in perfect food every time. Don't miss out! Order today!