

DISCRETE UNSOLVABILITY FOR THE INVERSE PROBLEM FOR ELECTRICAL NETWORKS

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ABSTRACT. For $n > 1$, we construct graphs X_n such that certain response matrices for X_n correspond to precisely n distinct conductivities on X_n . For a graph closely related to X_3 , we give a general algorithm for obtaining such a response matrix (i.e., one corresponding to precisely three distinct conductivities on the graph) from an arbitrary given response matrix. We (briefly) sketch the extension of this algorithm to general X_n .

1. BASIC DEFINITIONS AND CONVENTIONS

By a *graph with boundary* we mean an undirected graph X with vertex set V where

- (1) X is a connected, finite, and has no loops
- (2) disjoint subsets ∂V and V° of V are given, where ∂V is nonempty and $V = \partial V \cup V^\circ$
- (3) an identification of V with the first $|V|$ positive integers is given, such that all elements of ∂V precede all elements of V° .

(We will suppress part or all of this identification, whenever it is convenient to do so. Also, we will sometimes use an identification of V with the first $|V|$ non-negative integers, particularly when drawing graphs. To get from this identification to the implicit identification of V with the first $|V|$ positive integers, add 1.) Elements of ∂V are called *boundary nodes* (of X), and elements of V° are called *interior nodes*. When drawing graphs, boundary nodes will be represented by black dots, and interior nodes will be represented by black circles. As all graphs considered in this paper will be graphs with boundary, we will henceforth use the terms graph and graph with boundary interchangeably.

When we draw graphs, we may or may not draw the same node more than once. If a given node is drawn more than once, all instances of that node will be labeled with the same number. Thus, the two graphs shown in Figure 1 are in fact the same.

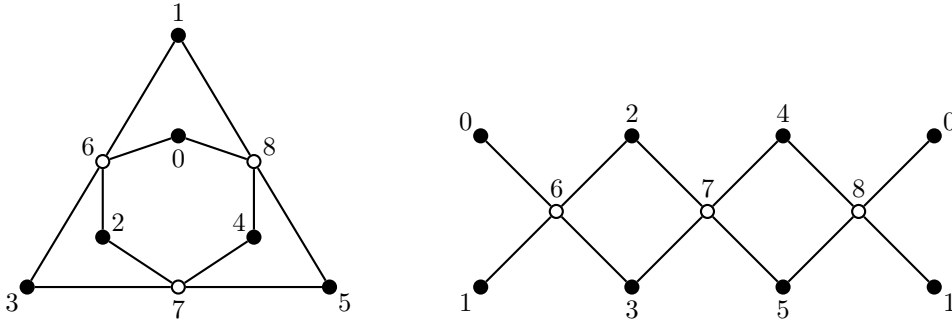


FIGURE 1. These two figures represent the same graph.

A *conductivity* on a graph is a positive function on the edges of that graph. If X is a graph and γ is a conductivity on X , the pair (X, γ) is called an *electrical network*.

Let (X, γ) be an electrical network. If i and j are distinct nodes in X , we define

$$(1) \quad \gamma_{i,j} = \sum_{\substack{\text{edges } e \\ \text{joining } i \text{ and } j}} \gamma(e),$$

where the empty sum is defined to be 0. If n is the number of nodes in X , we define the *Kirchhoff matrix* of the network (X, γ) to be the $n \times n$ matrix K given by

$$(2) \quad K_{i,j} = \begin{cases} \gamma_{i,j} & i \neq j \\ -\sum_{k \neq i} \gamma_{i,k} & i = j \end{cases}.$$

When multiple networks are under consideration, we will typically add the conductivity or the network as a subscript to K . Thus, $K = K_\gamma = K_{(X,\gamma)}$. If X has m boundary nodes, then K has the following useful block structure

$$(3) \quad K = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where A is $m \times m$ and C is $(n - m) \times (n - m)$. It is proven in [?] that C is invertible. (It is worth remarking that for the purposes of this paper, we only need to know that C is invertible when X lies in the class of graphs \underline{G} defined in Section 3, and for such X the invertibility of C is obvious, since C is diagonal by the first item in the definition of \underline{G} , and the diagonal entries of C are nonzero since graphs are, by our definition, connected.) We define the *response matrix* of (X, γ) to be the $m \times m$ matrix

$$(4) \quad \Lambda = A - BC^{-1}B^T = K/C,$$

that is, Λ is the Schur complement of C in K . Subscripting conventions for Λ are the same as those for K . We note that Λ may be obtained by performing Gaussian elimination on the last $n - m$ columns of K , and that the order in which we eliminate columns does not matter.

The term response matrix comes from the following ‘physical’ characterization of $\Lambda_{(X,\gamma)}$: if $\phi \in \mathbb{R}^n$, we can consider applying a potential to the boundary nodes of X (where the edges of X have conductances given by γ) whose value at a boundary node i is ϕ_i ; in this situation, for each i , the i th component of the vector $\Lambda_{(X,\gamma)}\phi$ is the (signed) current out of the boundary node i due to the applied voltage ϕ . See [?] for details.

If X is a graph, we will say that a matrix M is a Kirchhoff matrix for X if there is a conductivity γ on X with $K_{(X,\gamma)} = M$. If the graph X is understood and γ is a conductivity on X , we will call $K_{(X,\gamma)}$ the Kirchhoff matrix of γ . The same conventions apply for response matrices. The *inverse problem* is then as follows: given a graph X and a matrix L , find all conductivities on X with response matrix L .

It will be convenient to have some terminology regarding how ‘well-behaved’ a graph is with respect to the inverse problem. We say that a graph X is *recoverable* if *any* matrix L is the response matrix of at most one conductivity on X (so, if we know that L is a response matrix for X , we can at least theoretically ‘recover’ *the* conductivity on X with response matrix L from the information contained in L). If $n > 1$ is an integer, we say that a graph X is *n-to-1* if *some* matrix L is the response matrix of precisely n distinct conductivities on X . Finally, we say that a graph X is *∞ -to-1* (read ‘infinite to one’) if *any* response matrix L for X is the response matrix of infinitely many distinct conductivities on X .

Note that nothing we have said thus far precludes the possibility of a particular graph being *n-to-1* for more than one value of n , and nothing guarantees that a given graph will be either recoverable, *n-to-1* for some n , or *∞ -to-1*. However, it is immediate from the definitions that a graph cannot be more than one of recoverable, *n-to-1* for some n , and *∞ -to-1*. Also, it is worth remarking (though it may not be obvious from the point of view we have taken) that the properties ‘recoverable’, ‘*n-to-1*’, and ‘ *∞ -to-1*’ are independent of node-integer identification, in the sense that if X and X' are two graphs which differ only in the identification of their vertices with positive integers, then X is recoverable (resp. *n-to-1*, *∞ -to-1*) iff X' is recoverable (resp. *n-to-1*, *∞ -to-1*).

2. MOTIVATION

In this section, we sketch how the problem of finding n -to-1 graphs arose. When first considering the inverse problem, Morrow, Curtis, and Ingerman were interested in finding recoverable graphs. Work on this problem is documented at <http://math.washington.edu/~reu/>. A notable result in this direction is that a circular planar graph is recoverable iff it is critical. (Definitions of circular planar and critical (for circular planar graphs) can be found in [?].)

The search for recoverable graphs led naturally to consideration of non-recoverable graphs and ways in which a graph can fail to be recoverable. Certain graphs are obviously not recoverable: perhaps the simplest examples are the so-called series and parallel connections in Figure 2. Simple algebra and the definition of response matrix shows that these graphs are ∞ -to-1. Somewhat relatedly, the aforementioned recoverability result was strengthened by Jeff Giansiracusa, who showed that a non-critical circular planar graph is ∞ -to-1. Prior to this, Ernie Esser, when considering recoverability of so-called ‘annular networks’, discovered (by purely symbolic methods) a (rather simple) 2-to-1 graph, shown in Figure 1.



FIGURE 2. The parallel and series connections, respectively.

The existence of this single 2-to-1 graph (and some closely related 2^n -to-1 graphs) together with the fact that a large class of graphs (i.e., circular planar graphs) could not be n -to-1 led to the question of whether or not n -to-1 graphs existed for each n . (For a while, it was in fact conjectured that 2^n -to-1 graphs were the only possibilities.) The first significant progress on this problem was made by Ilya Grigoriev, who succeeded in constructing (albeit largely without proof) a 3-to-1 graph. This paper describes a modification and generalization of his approach, which works for arbitrary n .

3. PRELIMINARY NOTIONS

Denote by $\underline{\underline{G}}$ the space of graphs with the following properties:

- no two interior nodes are adjacent
- every interior node is adjacent to at least three boundary nodes
- no more than one edge joins a given interior node and boundary node.

We define $\underline{\underline{\Gamma}}$ to be the space of electrical networks whose underlying graph is in $\underline{\underline{G}}$.

3.1. Operation on Graphs. Throughout this subsection, X will be a fixed but arbitrary graph in $\underline{\underline{G}}$, n will be the number of nodes in X , and m the number of boundary nodes.

We now define an operation S from $\underline{\underline{G}}$ to itself. If $m = n$, then $SX = X$. Otherwise, SX is defined as follows. The nodes of SX are the nodes $1, \dots, n-1$ of X . If i and j are distinct nodes in SX and either i or j is not adjacent to n in X , then the edges between i and j in SX are the same as those in X . Otherwise, both i and j are adjacent to n in X , and the edges between i and j in SX are those in X together with an additional edge which we will denote e_{ij}^n . We declare the nodes $1, \dots, m$ in SX to be boundary nodes, and

the nodes $m+1, \dots, n-1$ to be interior nodes. It is clear that SX meets our requirements for a graph, and moreover that $SX \in \underline{\underline{G}}$.

The sequence $\{S^i X\}_{i=0}^\infty$ stabilizes at $i = n - m$. We call $S^{n-m}X$ the *star-K transformation of X* and define $X^* = S^{n-m}X$. Note that as X^* has no interior nodes, we have

$$(5) \quad K_{(X^*, \eta)} = \Lambda_{(X^*, \eta)}$$

for any conductivity η on X^* .

For $m < p \leq n$, let $A_p X$ be the set of nodes in X adjacent to p . By the definition of S , if $m < q \leq p \leq n$, then $\{e_{ij}^p\}_{i,j \in A_p X}$ is a collection of boundary-boundary edges in $S^{n-q+1}X$. Given $m < q \leq p \leq n$, we say that a conductivity δ on $S^{n-q+1}X$ satisfies the *quadrilateral rule* if for any $q \leq p \leq n$ and any distinct nodes i, j, k, l in $A_p X$ we have

$$(6) \quad \delta(e_{ij}^p)\delta(e_{kl}^p) = \delta(e_{ik}^p)\delta(e_{jl}^p).$$

This is equivalent to what we call the *triangle condition*: if i is a given node in $A_p X$, then the quantity

$$(7) \quad \frac{\delta(e_{ij}^n)\delta(e_{ik}^n)}{\delta(e_{jk}^n)}$$

is the same for any choice of j and k in $A_p X$ which are distinct from each other and from i . By convention, we say that any conductivity on X satisfies the quadrilateral rule.

We say that a Kirchhoff matrix K for X^* satisfies the *quadrilateral rule* if whenever $m < p \leq n$ and i, j, k, l are distinct nodes in $A_p X$ with a unique edge joining i to j , k to l , i to k and j to l in X^* , we have

$$(8) \quad K_{i,j}K_{k,l} = K_{i,k}K_{j,l}.$$

Note that if γ is a conductivity on X^* which satisfies the quadrilateral rule, then K_γ satisfies the quadrilateral rule. By (5), Lemma 3.2, and Corollary 3.3, any response matrix for X satisfies the quadrilateral rule as a Kirchhoff matrix for X^* .

3.2. Operation on Networks. We now extend S to an operation from $\underline{\underline{\Gamma}}$ to itself. Given $(X, \gamma) \in \underline{\underline{\Gamma}}$, we define a conductivity $S\gamma$ on SX as follows. Let

$$(9) \quad \sigma_n = \sum_{i \neq n} \gamma_{i,n},$$

where n is the number of nodes in X . If e is an edge in both SX and X , then we set

$$(10) \quad S\gamma(e) = \gamma(e).$$

If e is an edge in SX but not in X , then e is e_{ij}^n for some i and j (in particular, n is in fact an interior node of X), and we set

$$(11) \quad S\gamma(e_{ij}^n) = \frac{\gamma_{i,n}\gamma_{j,n}}{\sigma_n}.$$

Thus, we have $(SX, S\gamma) \in \underline{\underline{\Gamma}}$, and we set $S(X, \gamma) = (SX, S\gamma)$.

Note that $S\gamma$ satisfies the quadrilateral rule: if n is an interior node of X and i, j, k, l are distinct nodes in $A_n X$, we have

$$(12) \quad S\gamma(e_{ij}^n)S\gamma(e_{kl}^n) = \frac{\gamma_{i,n}\gamma_{j,n}}{\sigma_n} \cdot \frac{\gamma_{k,n}\gamma_{l,n}}{\sigma_n} = \frac{\gamma_{i,n}\gamma_{k,n}}{\sigma_n} \cdot \frac{\gamma_{j,n}\gamma_{l,n}}{\sigma_n} = S\gamma(e_{ik}^n)S\gamma(e_{jl}^n).$$

3.3. Some Basic Results. We now compile some useful properties of the map S .

Lemma 3.1. *If $X \in \underline{\underline{G}}$, then S defines a bijection from $\Gamma_X = \{\text{conductivities on } X\}$ to $\Gamma'_{SX} = \{\text{conductivities on } SX \text{ satisfying the quadrilateral rule}\}$.*

Proof. If X has no interior nodes, there is nothing to show, so assume that X has n nodes and that n is in fact an interior node of X .

First, we introduce some notation. If i and j are nodes in X , the expression $i \sim j$ means that i and j are adjacent in X . If i is a node in X which is adjacent to n , we will let e_i be the unique edge in X between i and n . If $\eta \in \Gamma'_{SX}$ is given and i is adjacent to n in X , then by the definition of $\underline{\underline{G}}$ there exist nodes j and k in X which are adjacent to n and distinct from each other and from i , and by (7) the quantity

$$(13) \quad \alpha_i^\eta = \sqrt{\frac{\eta(e_{ij}^n)\eta(e_{ik}^n)}{\eta(e_{jk}^n)}},$$

depends only on i . Observe that for distinct i and j , we have

$$(14) \quad \alpha_i^\eta \alpha_j^\eta = \sqrt{\frac{\eta(e_{ik}^n)\eta(e_{ij}^n)}{\eta(e_{kj}^n)}} \sqrt{\frac{\eta(e_{jk}^n)\eta(e_{ji}^n)}{\eta(e_{ki}^n)}} = \eta(e_{ij}^n),$$

where k is distinct from both i and j but is otherwise arbitrary.

Now, we define a map $T : \Gamma'_{SX} \rightarrow \Gamma_X$. Let $\eta \in \Gamma'_{SX}$ be given, and define a conductivity $T\eta$ on X as follows. If e is an edge in X which is also in SX , set

$$(15) \quad T\eta(e) = \eta(e).$$

Any other edge e in X is e_i for some i , and we define

$$(16) \quad T\eta(e_i) = \alpha_i^\eta \sum_{j \sim n} \alpha_j^\eta.$$

By (12), S defines a map from Γ_X to Γ'_{SX} . We claim that T is a two-sided inverse for S . Let $\delta \in \Gamma'_{SX}$ be given; we wish to show that $ST\delta = \delta$. Observe that

$$\begin{aligned} ST\delta(e_{ij}^n) &= \frac{T\delta(e_i)T\delta(e_j)}{\sum_{k \sim n} T\delta(e_k)} && \text{by (11)} \\ &= \frac{(\alpha_i^\delta \sum_{k \sim n} \alpha_k^\delta)(\alpha_j^\delta \sum_{l \sim n} \alpha_l^\delta)}{\sum_{k \sim n} (\alpha_k^\delta \sum_{l \sim n} \alpha_l^\delta)} && \text{by (16)} \\ &= \alpha_i^\delta \alpha_j^\delta && \text{after obvious cancellation} \\ &= \delta(e_{ij}^n) && \text{by (14)}. \end{aligned}$$

As δ and $ST\delta$ obviously agree on edges which are in both X and SX (by (10) and (15)), it follows that $ST\delta = \delta$.

Next, let $\gamma \in \Gamma_X$; we must show that $TS\gamma = \gamma$. We have

$$\begin{aligned}
 TS\gamma(e_i) &= \sum_{j \sim n} \alpha_i^{S\gamma} \alpha_j^{S\gamma} && \text{by (16)} \\
 &= \sum_{j \sim n} S\gamma(e_{ij}^n) && \text{by (14)} \\
 &= \sum_{j \sim n} \frac{\gamma(e_i)\gamma(e_j)}{\sum_{k \sim n} \gamma(e_k)} && \text{by (11)} \\
 &= \gamma(e_i) && \text{after obvious cancellation.}
 \end{aligned}$$

As γ and $TS\gamma$ agree on edges which are in both X and SX , the claim follows. \square

Lemma 3.2. *If $(X, \gamma) \in \underline{\Gamma}$ then $\Lambda_{(X, \gamma)} = \Lambda_{S(X, \gamma)}$.*

Proof. If X has no interior nodes, there is nothing to show. Otherwise, let n be the number of nodes in X , $K = K_{(X, \gamma)}$, and recall the definition of σ_n in (9). By the definition of S , we have

$$(17) \quad (K_{S(X, \gamma)})_{i,j} = S\gamma_{i,j} = \gamma_{i,j} + \frac{\gamma_{i,n}\gamma_{j,n}}{\sigma_n} = K_{i,j} + \frac{K_{i,n}K_{j,n}}{\sigma_n}$$

for any $i \neq j$ (where the quantity $\frac{\gamma_{i,n}\gamma_{j,n}}{\sigma_n}$ may very well be zero).

Consider the matrix K' obtained from K by adding multiples of the n th row of K (with nonzero n th entry $K_{n,n} = -\sigma_n$) to the first $n - 1$ rows to reduce the n th column of K to $-\sigma_n \mathbf{e}_n$ (where \mathbf{e}_n is the n th standard basis vector), and then removing the n th row and column. By definition, K' satisfies

$$(18) \quad K'_{i,j} = K_{i,j} + \frac{K_{i,n}K_{j,n}}{\sigma_n}$$

for all i and j .

Thus, we have $K'_{i,j} = (K_{S(X, \gamma)})_{i,j}$ for $i \neq j$. By the definition of Kirchhoff matrix, $K_{S(X, \gamma)}$ has row sum zero. Note that K' also has row sum zero, as the Kirchhoff matrix K from which K' was obtained has row sum zero. Thus, we have $K' = K_{S(X, \gamma)}$. That $\Lambda_{(X, \gamma)} = \Lambda_{S(X, \gamma)}$ is now obvious, as K' was obtained from K by performing Gaussian elimination on the column corresponding to the last interior node of X and then removing this column and the corresponding row. \square

Corollary 3.3. *If $X \in \underline{G}$ and L is a given matrix, then (a power of) S induces a bijection from $\{(X, \gamma) : \Lambda_{(X, \gamma)} = \overline{L}\}$ to $\{(X^*, \gamma) : K_{(X^*, \gamma)} = L \text{ and } \gamma \text{ satisfies the quadrilateral rule}\}$.*

Proof. Repeated applications of Lemma 3.1 together with the definition of S show that a suitable power of S induces a bijection from $\{\text{conductivities on } X\}$ to $\{\text{conductivities on } X^* \text{ satisfying the quadrilateral rule}\}$. By Lemma 3.2, this bijection preserves response matrices. The result now follows from (5). \square

4. PRELIMINARIES

Let X denote the graph in Figure 3, and X^* its star- K transformation in Figure 4. By convention, when drawing the star- K transformation of a given graph, we position edges in a manner which ‘respects incidence’ in the original graph. Thus, for example, in the drawing of X^* in Figure 4, the edges e_0 and e_1 are, in the notation of our definition of S , $e_{0,1}^{14}$ and $e_{12,13}^{14}$, respectively, whereas e_2 and e_3 are $e_{12,13}^{15}$ and $e_{5,6}^{15}$, respectively. (Note that we could equally well have interchanged e_1 and e_2 , having them denote $e_{12,13}^{15}$ and

$e_{12,13}^{14}$ respectively, while still drawing and labeling X^* just as in Figure 4, but this would obviously be somewhat pathological.) It will be important to keep this convention in mind while making sense of the rest of this paper.

Define the following functions, where K is any Kirchhoff matrix for X^* which satisfies the quadrilateral rule and x is a real parameter:

$$\begin{aligned}
(19) \quad & f_0(K; x) = x \\
(20) \quad & f_1(K; x) = \frac{K_{0,12}K_{1,13}}{f_0(K; x)} \\
(21) \quad & f_2(K; x) = K_{12,13} - f_1(K; x) \\
(22) \quad & f_3(K; x) = \frac{K_{5,12}K_{6,13}}{f_2(K; x)} \\
(23) \quad & f_4(K; x) = K_{5,6} - f_3(K; x) \\
(24) \quad & f_5(K; x) = \frac{K_{4,7}}{K_{6,7}}f_4(K; x) \\
(25) \quad & f_6(K; x) = K_{4,5} - f_5(K; x) \\
(26) \quad & f_7(K; x) = \frac{K_{2,4}K_{3,5}}{f_6(K; x)} \\
(27) \quad & f_8(K; x) = K_{2,3} - f_7(K; x) \\
(28) \quad & f_9(K; x) = \frac{K_{0,2}K_{1,3}}{f_8(K; x)} \\
(29) \quad & f_{10}(K; x) = \frac{K_{4,7}}{K_{5,7}}f_4(K; x) \\
(30) \quad & f_{11}(K; x) = K_{4,6} - f_{10}(K; x) \\
(31) \quad & f_{12}(K; x) = \frac{K_{4,8}K_{6,9}}{f_{11}(K; x)} \\
(32) \quad & f_{13}(K; x) = K_{8,9} - f_{12}(K; x) \\
(33) \quad & f_{14}(K; x) = \frac{K_{0,8}K_{9,10}}{f_{13}(K; x)} \\
(34) \quad & f_{15}(K; x) = K_{0,10} - f_{14}(K; x) \\
(35) \quad & f_{16}(K; x) = \frac{K_{1,11}}{K_{0,11}}f_{15}(K; x) \\
(36) \quad & f_{17}(K; x) = K_{1,10} - f_{16}(K; x) \\
(37) \quad & f_{18}(K; x) = \frac{K_{1,11}}{K_{10,11}}f_{15}(K; x) \\
(38) \quad & \sigma(K; x) = x + f_{18}(K; x) \\
(39) \quad & \Sigma(K; x) = \sigma(K; x) + f_9(K; x)
\end{aligned}$$

It is trivial to verify that each $f_j(K; x)$ is a linear fractional transformation of x for fixed K , and that the sign of $\partial_x f_j(K; x)$ is independent of K . We note here that

$$(40) \quad f_7, f_9, f_{12}, \text{ and } f_{14} \text{ have positive } x \text{ derivative,}$$

while

$$(41) \quad f_8, f_{13}, f_{15}, f_{16}, \text{ and } f_{18} \text{ have negative } x \text{ derivative.}$$

Given a Kirchhoff matrix K for X^* which satisfies the quadrilateral rule, we define

$$(42) \quad \chi(K) \equiv \{x \in \mathbb{R} : f_j(K; x) > 0 \text{ for all } j\}.$$

As the functions $f_j(K; x)$ are linear fractional transformations of x , it is clear that $\chi(K)$ is open. It is also true that $\chi(K)$ is an interval, though we will not need this fact.

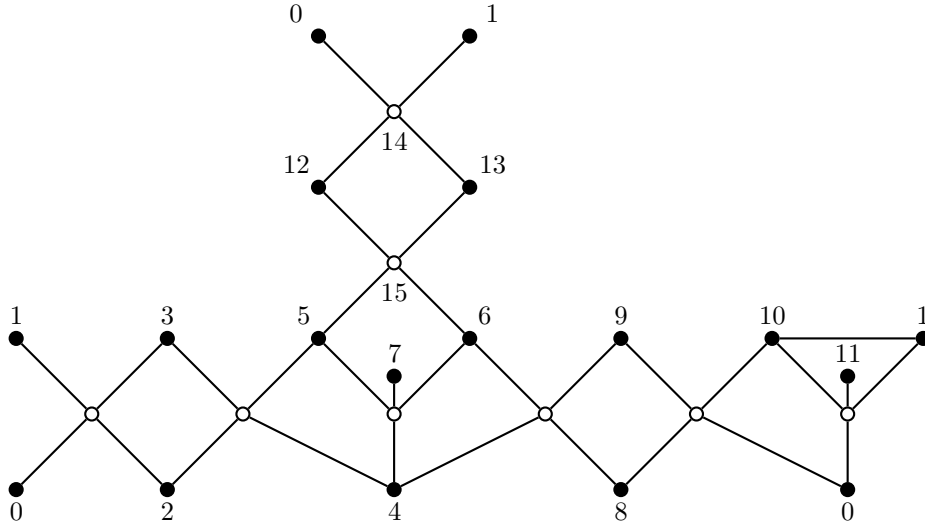


FIGURE 3. The graph X .

Now, we make a few observations which will be useful later. Suppose K is any Kirchhoff matrix for X^* which satisfies the quadrilateral rule. Suppose γ^* is a conductivity on X^* satisfying the quadrilateral rule with $K_{\gamma^*} = K$. By the construction of the functions f_j and the hypotheses on γ^* , we have $\gamma^*(e_j) = f_j(K; \gamma^*(e_0))$ for all j . In particular, as γ^* is a conductivity, we have $f_j(K; \gamma^*(e_0)) > 0$ for all j , i.e., $\gamma^*(e_0) \in \chi(K)$. As $K_{\gamma^*} = K$, we also have $\Sigma(K; \gamma^*(e_0)) = K_{0,1}$. In other words, $\gamma^*(e_0)$ is a number $a \in \chi(K)$ such that $\Sigma(K; a) = K_{0,1}$. Note that distinct conductivities γ^* satisfying the quadrilateral rule and having Kirchhoff matrix K must assume distinct values on the edge e_0 , as such a conductivity γ^* is determined by its value on e_0 . (For, we have seen that $\gamma^*(e_j) = f_j(K; \gamma^*(e_0))$, and the values of γ^* on edges other than the e_j can be read off directly from K : if e is an edge in X^* which is distinct from all the e_j , then inspection of X^* shows that e is the unique edge joining some nodes i and k in X^* , and if we denote this edge e by e_{ik} , then we have $\gamma^*(e_{ik}) = K_{i,k}$.) If, conversely, $b \in \chi(K)$ is given and $\Sigma(K; b) = K_{0,1}$, then we may define a conductivity γ^* on X^* by setting $\gamma^*(e_j) = f_j(K; b)$ and $\gamma^*(e_{ik}) = K_{i,k}$ (with the e_{ik} as above). It follows immediately from the definition of the f_j , the hypotheses on b , and the hypothesis that K satisfies the quadrilateral rule that γ^* satisfies the quadrilateral rule and $K_{\gamma^*} = K$. It is clear that distinct values of $b \in \chi(K)$ with $\Sigma(K; b) = K_{0,1}$ give rise to distinct conductivities γ^* . Thus, there is a bijection between conductivities on X^* satisfying the quadrilateral rule with Kirchhoff matrix K and numbers $c \in \chi(K)$ such that $\Sigma(K; c) = K_{0,1}$.

By Corollary 3.3, in order to show that X is 3-to-1, it suffices to find a Kirchhoff matrix K for X^* which satisfies the quadrilateral rule and has precisely three distinct values of $c \in \chi(K)$ with $\Sigma(K; c) = K_{0,1}$. Note that if K determines functions $f_9(K; x)$ and $f_{18}(K; x)$ having singularities as functions of x , then there are at most three solutions c to $\Sigma(K; c) = K_{0,1}$ (just multiply through by the denominators of both $f_9(K; x)$ and $f_{18}(K; x)$, and consider the resulting genuine cubic equation). Therefore, it suffices to find a Kirchhoff matrix K satisfying the quadrilateral rule which admits at least three solutions $c \in \chi(K)$ to $\Sigma(K; c) = K_{0,1}$ and which determines functions $f_9(K; x)$ and $f_{18}(K; x)$ having singularities. We provide a general way of constructing such K in the next section.

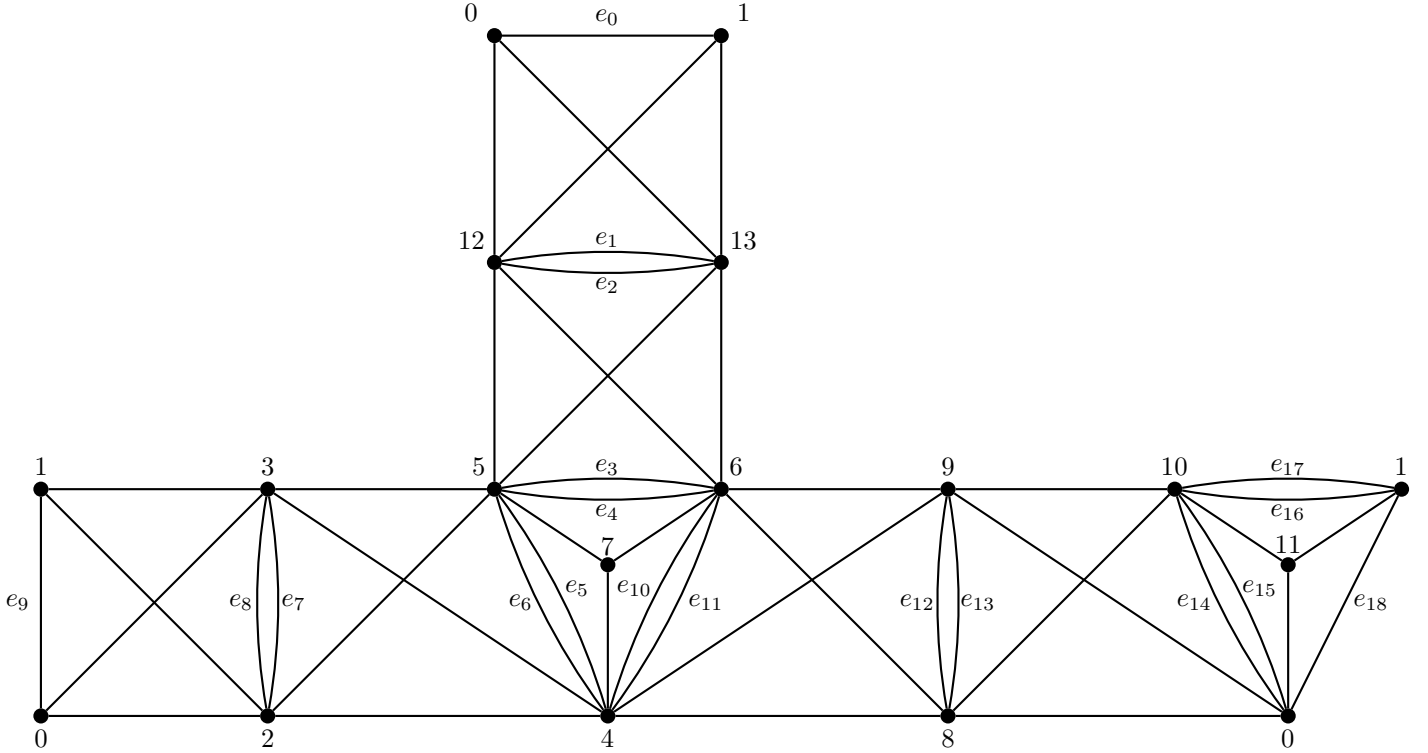
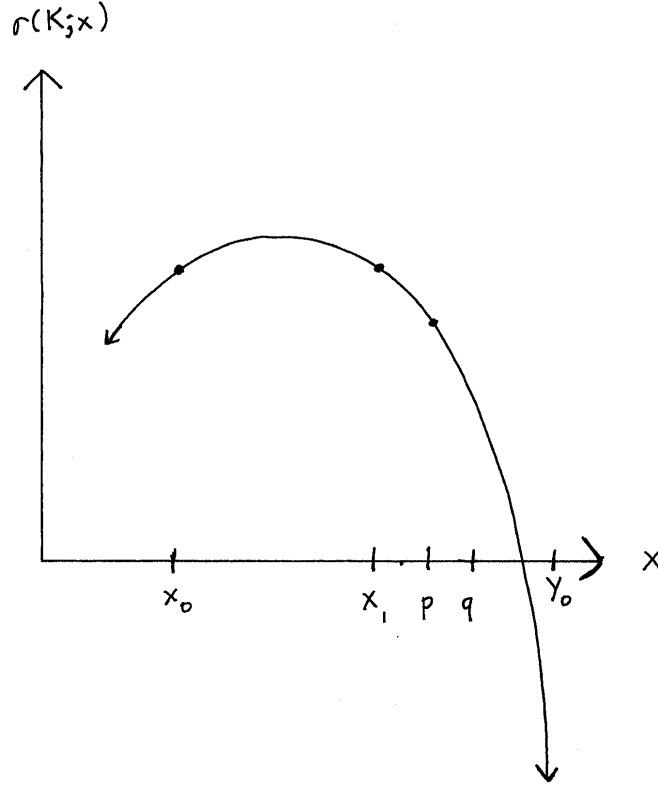


FIGURE 4. The star- K transformation of X .

5. OUTLINE

It is easy to get lost in the details of this method, so we provide a brief overview (lacking complete justifications) of our approach here. Our goal is to produce a Kirchhoff matrix K for X^* , satisfying the quadrilateral rule, determining functions $f_9(K; x)$ and $f_{18}(K; x)$ having singularities, with the property that $\Sigma(K; x) = K_{0,1}$ has three distinct solutions in $\chi(K)$. We will accomplish this by producing a Kirchhoff matrix K for X^* , satisfying the quadrilateral rule, with the property that there exist (positive) numbers $x_0 < x_1 < p < q < y_0$ such that

- (i) $[x_0, q] \subseteq \chi(K)$
- (ii) $\sigma(K; x)$ is singular at y_0 (i.e., $f_{18}(K; x)$ is singular at y_0)


 FIGURE 5. The desired behavior of $\sigma(K; x)$.

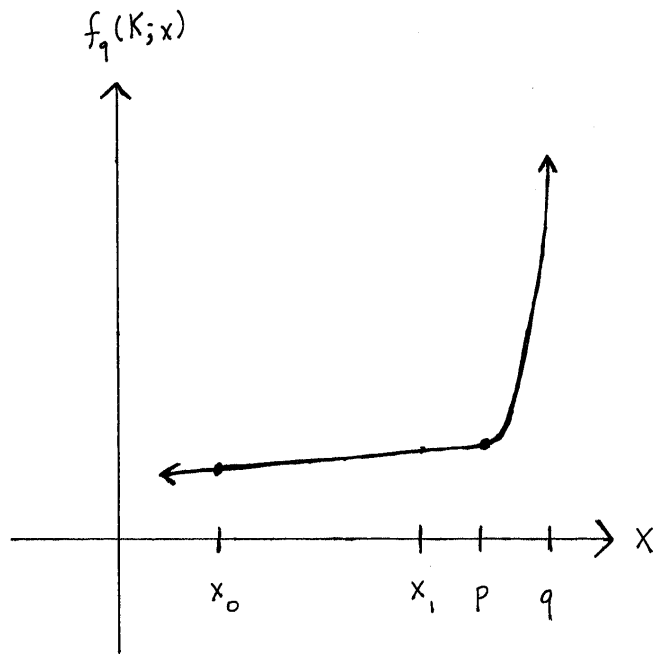
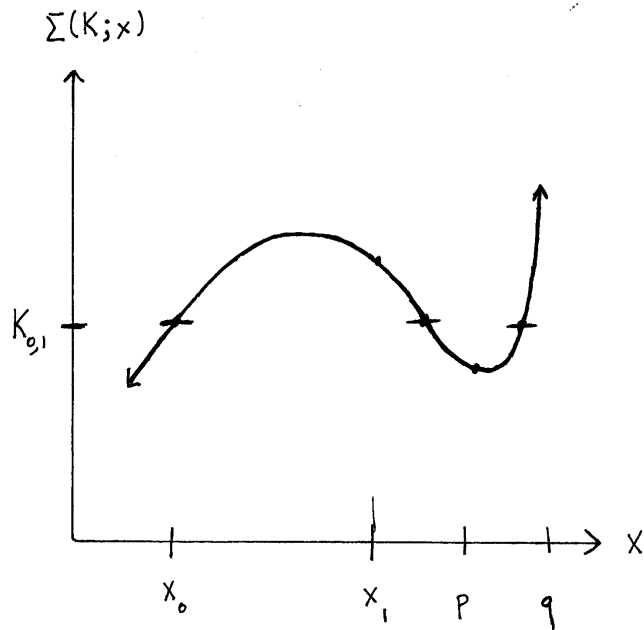
- (iii) $f_9(K; x)$ is singular at q
- (iv) $\sigma(K; x_1) = \sigma(K; x_0)$
- (v) $f_9(K; p) - f_9(K; x_0) < \sigma(K; x_1) - \sigma(K; p)$

The idea is that such a K determines functions $\sigma(K; x)$ and $f_9(K; x)$ as depicted in Figures 5 and 6, and that when added together, these yield a function $\Sigma(K; x)$ as in Figure 7, which has all the necessary properties (after we redefine $K_{0,1}$, if necessary).

We will produce a K satisfying conditions (i)-(v) above in the following way. First, let K^1 be any Kirchhoff matrix for X^* which satisfies the quadrilateral rule (for example, any response matrix for X will do). Choose any numbers $x_0 < y_0$ with $[x_0, y_0] \subseteq \chi(K^1)$. We first focus on the behavior of σ .

From K^1 we produce a Kirchhoff matrix K^2 for X^* which also satisfies the quadrilateral rule, and which has the property that $\sigma(K^2; x)$ is singular at y_0 (so, K^2 satisfies condition (ii) above). By the definition of σ and the functions f_j , we have the following equivalences:

$$\begin{aligned}
 \sigma(K^2; x) \text{ is singular at } y_0 &\text{ iff } f_{18}(K^2; x) \text{ is singular at } y_0 \\
 &\text{ iff } f_{15}(K^2; x) \text{ is singular at } y_0 \\
 &\text{ iff } f_{13}(K^2; y_0) = 0 \\
 &\text{ iff } f_{12}(K^2; y_0) = K_{8,9}^2.
 \end{aligned}$$

FIGURE 6. The desired behavior of $f_0(K; x)$.FIGURE 7. The desired behavior of $\Sigma(K; x)$.

A moment's thought shows that we may take K^2 to be the (unique) Kirchhoff matrix for X^* agreeing with K^1 below the diagonal except that $K_{8,9}^2 = f_{12}(K^1; y_0)$.

Now, we produce a Kirchhoff matrix K^3 for X^* which also satisfies the quadrilateral rule, and which has the properties that $\partial_x \sigma(K^3; x_0) > 0$ and $\sigma(K^3; x)$ is singular at y_0 . By the definition of σ and f_{18} , and our observation in (41) that f_{15} and f_{18} have negative x derivative, we have the following equivalences:

$$\partial_x \sigma(K^3; x_0) > 0 \text{ iff } \partial_x f_{18}(K^3; x_0) > -1 \text{ iff } K_{10,11}^3 > K_{1,11}^3 |\partial_x f_{15}(K^3; x_0)|.$$

We can in fact take K^3 to be the unique Kirchhoff matrix for X^* agreeing with K^2 below the diagonal except that $K_{10,11}^3 = 2K_{1,11}^2 |\partial_x f_{15}(K^2; x_0)|$; in particular, modifying K^2 to obtain K^3 in this way preserves the singularity of σ at y_0 .

Note that $\sigma(K^3; x)$ approaches $-\infty$ as x approaches y_0 from the left, as the same is true of $f_{18}(K^3; x)$ by (41). Thus, as $\sigma(K^3; x)$ is increasing at x_0 by construction, the Intermediate Value Theorem guarantees a point $x_1 \in (x_0, y_0)$ with $\sigma(K^3; x_1) = \sigma(K^3; x_0)$. Thus, K^3 satisfies conditions (ii) and (iv) above.

Having obtained K^3 from K^1 as above, the following are facts:

- $f_j(K^3; x) = f_j(K^1; x)$ for $j < 13$
- $f_{13}(K^3; x)$ and $f_{14}(K^3; x)$ are positive (hence continuous) on $[x_0, y_0]$.

In particular, $f_{14}(K^3; x)$ is bounded on $[x_0, x_1]$.

Now, we can define a Kirchhoff matrix K^4 for X^* which satisfies the quadrilateral rule, such that $\sigma(K^4; x)$ differs from $\sigma(K^3; x)$ by an additive constant and $[x_0, x_1] \subseteq \chi(K^4)$. By the two bulleted items above, we have $f_j(K^3; x) > 0$ on $[x_0, x_1]$ for $j < 15$; of course, we also want this condition to hold with K^3 replaced by K^4 , and the easiest way to guarantee that it will is to avoid modifying K^3 (in the process of obtaining K^4) in ways which alter the f_j for $j < 15$. With this in mind, we focus on $f_j(K^4; x)$ for $j > 14$. By (34), we have $f_{15}(K^4; x) = K_{0,10}^4 - f_{14}(K^4; x)$; provided $f_{14}(K^4; x) = f_{14}(K^3; x)$, we therefore have

$$f_{15}(K^4; x) > 0 \text{ on } [x_0, x_1] \text{ iff } K_{0,10}^4 > \sup_{x \in [x_0, x_1]} f_{14}(K^3; x).$$

Note that for all x , the signs of $f_{15}(K^4; x)$, $f_{16}(K^4; x)$, and $f_{18}(K^4; x)$ will agree by (35) and (37). Thus, we consider $f_{17}(K^4; x)$. By (34)-(36), we have

$$f_{17}(K^4; x) = K_{1,10}^4 - \frac{K_{1,11}^4}{K_{0,11}^4} (K_{0,10}^4 - f_{14}(K^4; x)).$$

Assuming as above that $f_{14}(K^4; x) = f_{14}(K^3; x)$, and additionally that $K_{1,11}^4 = K_{1,11}^3$ and $K_{0,11}^4 = K_{0,11}^3$, it follows that

$$f_{17}(K^4; x) > 0 \text{ on } [x_0, x_1] \text{ iff } K_{1,10}^4 > \sup_{x \in [x_0, x_1]} \frac{K_{1,11}^3}{K_{0,11}^3} (K_{0,10}^4 - f_{14}(K^3; x)).$$

It turns out that the above equivalences suggest a suitable K^4 : we can take K^4 to be the unique Kirchhoff matrix for X^* agreeing with K^3 below the diagonal except that

$$K_{0,10}^4 = 1 + \sup_{x \in [x_0, x_1]} f_{14}(K^3; x)$$

and

$$K_{1,10}^4 = 1 + \sup_{x \in [x_0, x_1]} \frac{K_{1,11}^3}{K_{0,11}^3} (K_{0,10}^4 - f_{14}(K^3; x)).$$

That $\sigma(K^4; x)$ differs from $\sigma(K^3; x)$ by an additive constant follows as

$$f_{18}(K^4; x) = \frac{K_{1,11}^4}{K_{10,11}^4} (K_{0,10}^4 - f_{14}(K^4; x))$$

by (34) and (37), and

$$\frac{K_{1,11}^4}{K_{10,11}^4} (K_{0,10}^4 - f_{14}(K^4; x)) = \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - f_{14}(K^3; x))$$

as $K_{1,11}^4 = K_{1,11}^3$, $K_{10,11}^4 = K_{10,11}^3$, and $f_{14}(K^4; x) = f_{14}(K^3; x)$, where the right hand side of the previous displayed equation differs from $f_{18}(K^3; x)$ by an additive constant (namely $\frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - K_{0,10}^3)$).

One can show that $\partial_x \sigma(K^4; x_1) < 0$. As $[x_0, x_1] \subseteq \chi(K^4)$ and $\chi(K^4)$ is open, it follows that there exist positive numbers p, q, r with

$$\begin{aligned} x_1 &< p < q < y_0, \\ [x_0, q] &\subseteq \chi(K^4), \end{aligned}$$

and

$$\sigma(K^4; x_1) = \sigma(K^4; p) + r.$$

At this point, the function $\sigma(K^4; x)$ has the form of Figure 5, and K^4 satisfies conditions (i), (ii), and (iv).

Finally, we modify the behavior of f_9 . Our goal is to produce a Kirchhoff matrix K^5 for X^* which satisfies the quadrilateral rule, such that $\sigma(K^5; x) = \sigma(K^4; x)$ and $f_9(K^5; x)$ behaves as in Figure 6, with $[x_0, q] \subseteq \chi(K^5)$. First of all, we need $f_9(K^5; x)$ to be singular at q ; by the definition of the functions f_j , we have

$$f_9(K^5; x) \text{ is singular at } q \text{ iff } f_8(K^5; q) = 0 \text{ iff } f_7(K^5; q) = K_{2,3}^5.$$

This suggests we set

$$(43) \quad K_{2,3}^5 = f_7(K^4; q)$$

and take care to guarantee that $f_7(K^5; x) = f_7(K^4; x)$.

We also want $f_9(K^5; p) - f_9(K^5; x_0) < \sigma(K^5; x_1) - \sigma(K^5; p)$. Provided that $f_9(K^5; x)$ is singular at q and $\sigma(K^5; x) = \sigma(K^4; x)$, this condition will hold if $\partial_x f_9(K^5; p) < \frac{r}{p-x_0}$. (To see that this is true, simply recall the definition $r = \sigma(K^4; x_1) - \sigma(K^4; p)$ and the fact that $f_9(K^5; x)$ will be a linear fractional transformation of x , so if $f_9(K^5; x)$ is singular at q then $|\partial_x f_9(K^5; x)|$ is a decreasing function of $|x - q|$.) By (27) and (28), we have

$$f_9(K^5; x) = \frac{K_{0,2}^5 K_{1,3}^5}{K_{2,3}^5 - f_7(K^5; x)}.$$

Anticipating our choice of $K_{2,3}^5$ above, and assuming that $f_7(K^5; x) = f_7(K^4; x)$, the Chain Rule suggests we take

$$(44) \quad K_{0,2}^5 = K_{1,3}^5 = K_{1,2}^5 = K_{0,3}^5 = \sqrt{\frac{r(f_7(K^4; q) - f_7(K^4; p))^2}{2(p-x_0)\partial_x f_7(K^4; p)}},$$

which, provided the step from K^4 to K^5 does not change f_9 in any yet unconsidered ways, will give $\partial_x f_9(K^5; p) = \frac{r}{2(p-x_0)}$.

The last modification we need to make simply involves getting the right value for $K_{0,1}^5$; a natural choice seems to be stipulating that $\Sigma(K^5; x_0) = K_{0,1}^5$, which holds if we define K^5

to be the unique Kirchhoff matrix for X^* agreeing with K^4 below the diagonal except that equations (43) and (44) hold, and

$$K_{0,1}^5 = \frac{K_{0,2}^5 K_{1,3}^5}{f_7(K^4; q) - f_7(K^4; x_0)} + \sigma(K^4; x_0).$$

The matrix K^5 now satisfies conditions (i)-(v) and defines a function $\Sigma(K^5; x)$ as in Figure 7.

6. DETAILS

Let K^1 be a response matrix for X . Choose any (necessarily positive) numbers $x_0 < y_0$ with

$$(45) \quad [x_0, y_0] \subseteq \chi(K^1).$$

Define a new Kirchhoff matrix K^2 for X^* which agrees with K^1 below the diagonal except that

$$(46) \quad K_{8,9}^2 = f_{12}(K^1, y_0).$$

By (45), the right hand side of (46) is positive, so this definition makes sense. It is clear that K^2 satisfies the quadrilateral rule, as K^1 does.

We have

$$(47) \quad f_j(K^2; x) = f_j(K^1; x) \text{ for } j < 13,$$

as we have changed only a few entries in K^1 to obtain K^2 .

Note that (32), (46), and imply that

$$(48) \quad f_{13}(K^2; y_0) = 0,$$

so that

$$(49) \quad f_{15}(K^2; x) \text{ is singular at } y_0$$

by (33) and (34). In particular,

$$(50) \quad f_{15}(K^2; x) \text{ is differentiable at } x_0.$$

Observe that by (40) and (45), we have

$$(51) \quad f_{12}(K^1; y_0) > f_{12}(K^1; x_0),$$

so by (32), (46), and (6), we have

$$(52) \quad f_{13}(K^2; x_0) > 0.$$

Together with (41) and (48), this shows that $f_{13}(K^2; x)$ is nonsingular on $[x_0, y_0]$, and thus is positive on $[x_0, y_0]$. By (33), the sign of $f_{14}(K^2; x)$ is the same as the sign of $f_{13}(K^2; x)$, so we have shown that

$$(53) \quad f_{13}(K^2; x) \text{ and } f_{14}(K^2; x) \text{ are positive on } [x_0, y_0].$$

Now let K^3 be the Kirchhoff matrix for X^* which agrees with K^2 below the diagonal except that

$$(54) \quad K_{10,11}^3 = 2|\partial_x f_{15}(K^2; x_0)|K_{1,11}^2.$$

(The multiplicative 2 in (54) could be replaced by any number strictly larger than 1.) By (50), the right hand side of (54) is well-defined, and thus is positive, so that K^3 is indeed a Kirchhoff matrix. Also, K^3 satisfies the quadrilateral rule, as K^2 does.

Note that

$$(55) \quad K_{1,11}^2 = K_{1,11}^3.$$

As K^3 and K^2 agree at most of their entries, it is easy to see that

$$(56) \quad f_j(K^3; x) = f_j(K^2; x) \text{ for } j < 18.$$

By (37), (55), and (56), we have

$$(57) \quad f_{18}(K^3; x) = \frac{K_{1,11}^3}{K_{10,11}^3} f_{15}(K^3; x) = \frac{K_{1,11}^2}{K_{10,11}^3} f_{15}(K^2; x).$$

Thus,

$$(58) \quad f_{18}(K^3; x) \text{ is singular at } y_0$$

by (49) and (57). In particular, $f_{18}(K^3; x)$ is differentiable at x_0 , so we have

$$\begin{aligned} \partial_x f_{18}(K^3; x_0) &= \frac{K_{1,11}^2}{K_{10,11}^3} \partial_x f_{15}(K^2; x_0) && \text{by (57)} \\ &= \frac{1}{2} \frac{\partial_x f_{15}(K^2; x_0)}{|\partial_x f_{15}(K^2; x_0)|} && \text{by (54)} \\ &= -\frac{1}{2} && \text{by (41)}. \end{aligned}$$

Thus,

$$(59) \quad \partial_x \sigma(K^3; x_0) = \frac{1}{2}$$

by (38). Therefore, as x is bounded on $[x_0, y_0]$ while $f_{18}(K^3; x)$ is continuous on $[x_0, y_0]$ by (58) and $f_{18}(K^3; x) \rightarrow -\infty$ as $x \rightarrow y_0^-$ by (41) and (58), it follows from (38), (59), and the Intermediate Value Theorem that there exists

$$(60) \quad x_1 \in (x_0, y_0)$$

with

$$(61) \quad \sigma(K^3; x_0) = \sigma(K^3; x_1).$$

We also have that

$$(62) \quad f_{13}(K^3; x) \text{ and } f_{14}(K^3; x) \text{ are positive on } [x_0, y_0]$$

by (53) and (56).

Now, let K^4 be the Kirchhoff matrix for X^* agreeing with K^3 below the diagonal except that

$$(63) \quad K_{0,10}^4 = 1 + \sup_{x \in [x_0, x_1]} f_{14}(K^3; x)$$

and

$$(64) \quad K_{1,10}^4 = 1 + \sup_{x \in [x_0, x_1]} \frac{K_{1,11}^3}{K_{0,11}^3} (K_{0,10}^4 - f_{14}(K^3; x)).$$

(The additive 1s in (63) and (64) could be replaced by any positive numbers.) As $f_{14}(K^3; x)$ is positive on $[x_0, x_1]$ by (60) and (62), it is continuous there, and thus the supremum in (63) is finite, and also positive, so that (63) is positive. It follows that (64) is finite and positive

as well, so that K^4 is in fact a Kirchhoff matrix for X^* . As K^3 satisfies the quadrilateral rule, so too does K^4 . Note that

$$(65) \quad K_{1,11}^3 = K_{1,11}^4, \quad K_{0,11}^3 = K_{0,11}^4, \quad \text{and} \quad K_{10,11}^3 = K_{10,11}^4.$$

We claim that

$$(66) \quad [x_0, x_1] \subseteq \chi(K^4).$$

Note first that

$$(67) \quad f_j(K^4; x) = f_j(K^3; x) \text{ for } j < 15,$$

as K^3 and K^4 agree at most of their entries. Thus, from (34), we have

$$(68) \quad f_{15}(K^4; x) = K_{0,10}^4 - f_{14}(K^4; x) = K_{0,10}^4 - f_{14}(K^3; x).$$

In particular,

$$(69) \quad f_{15}(K^4; x) > 0 \text{ for } x \in [x_0, x_1]$$

by (63). It follows from (35) and (37) that

$$(70) \quad f_{16}(K^4; x) > 0 \text{ and } f_{18}(K^4; x) > 0 \text{ for } x \in [x_0, x_1].$$

From (35), (65), and (68), we obtain

$$(71) \quad f_{16}(K^4; x) = \frac{K_{1,11}^3}{K_{0,11}^3} (K_{0,10}^4 - f_{14}(K^3; x)).$$

Thus, by (64), we have

$$(72) \quad K_{1,10}^4 = 1 + \sup_{x \in [x_0, x_1]} f_{16}(K^4; x).$$

By (36), we then have

$$(73) \quad f_{17}(K^4; x) > 0 \text{ for } x \in [x_0, x_1].$$

We can now establish (66), i.e., that every $f_j(K^4; x)$ is positive on $[x_0, x_1]$. For $j < 13$, this follows from (45), (6), (56), (60), and (67). (Indeed, $f_j(K^4; x) = f_j(K^1; x)$ for $j < 13$.) For $j = 13, 14$, this follows from (62) and (67). For $j > 14$, this is precisely (69), (70), and (73).

We also claim that $\sigma(K^4; x_1) = \sigma(K^4; x_0)$. Observe that

$$\begin{aligned} f_{18}(K^4; x) &= \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - f_{14}(K^3; x)) && \text{by (37), (65), (68)} \\ &= \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^3 - f_{14}(K^3; x) + K_{0,10}^4 - K_{0,10}^3) \\ &= \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^3 - f_{14}(K^3; x)) + \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - K_{0,10}^3) \\ &= f_{18}(K^3; x) + \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - K_{0,10}^3) && \text{by (34), (37).} \end{aligned}$$

In other words,

$$(74) \quad f_{18}(K^4; x) = f_{18}(K^3; x) + c_0,$$

where

$$(75) \quad c_0 = \frac{K_{1,11}^3}{K_{10,11}^3} (K_{0,10}^4 - K_{0,10}^3)$$

is a constant. By (38), we have $\sigma(K^4; x) = \sigma(K^3; x) + c_0$. By (61), we therefore have

$$(76) \quad \sigma(K^4; x_0) = \sigma(K^4; x_1).$$

By (58) and (74), we have that

$$(77) \quad f_{18}(K^4; x) \text{ is singular at } y_0.$$

Thus, $\partial_x \sigma(K^4; x_1) \neq 0$ (for otherwise there are three solutions, counting multiplicity, to the quadratic equation associated to $\sigma(K^4; x) = \sigma(K^4; x_0)$ by clearing the denominator of $f_{18}(K^4; x)$). It follows that

$$(78) \quad \partial_x \sigma(K^4; x_1) < 0,$$

considering (38), (41), (60), (76), and (77). Thus, (66) and (78) show that there exist positive numbers p, q, r with

$$(79) \quad x_1 < p < q < y_0,$$

$$(80) \quad \sigma(K^4; x_1) = \sigma(K^4; p) + r,$$

and

$$(81) \quad [x_0, q] \subseteq \chi(K^4).$$

Now, we construct a new Kirchoff matrix K^5 for X^* such that K^5 agrees with K^4 below the diagonal except that

$$(82) \quad K_{2,3}^5 = f_7(K^4; q),$$

$$(83) \quad K_{0,2}^5 = K_{1,3}^5 = K_{1,2}^5 = K_{0,3}^5 = \sqrt{\frac{r(f_7(K^4; q) - f_7(K^4; p))^2}{2(p - x_0)\partial_x f_7(K^4; p)}},$$

and

$$(84) \quad K_{0,1}^5 = \frac{K_{0,2}^5 K_{1,3}^5}{f_7(K^4; q) - f_7(K^4; x_0)} + \sigma(K^4; x_0).$$

(The 2 in the denominator in (83) could be replaced by any number strictly larger than 1.) By (60), (79) and (81), $f_7(K^4; x)$ is differentiable at p , so by (40), the right hand side of (83) makes sense as a positive number. By (40) and (81), we have

$$(85) \quad f_7(K^4; q) > f_7(K^4; x_0) > 0.$$

Thus the right hand side of (82) is positive, and the same is true of (84) by (38), (81), and (85). It follows that K^5 is in fact a Kirchoff matrix for X^* . Also, by (83) and the fact that K^4 satisfies the quadrilateral rule, we have that K^5 satisfies the quadrilateral rule.

Note that

$$(86) \quad f_j(K^5; x) = f_j(K^4; x) \text{ for } j \neq 8, 9,$$

as K^4 and K^5 agree at most of their entries. Thus, by (27) and (82) we have

$$(87) \quad f_8(K^5; x) = K_{2,3}^5 - f_7(K^5; x) = f_7(K^4; q) - f_7(K^4; x).$$

Next, by (38) and (86), we have

$$(88) \quad \sigma(K^5; x) = \sigma(K^4; x).$$

Furthermore,

$$(89) \quad f_9(K^5; x) = \frac{K_{0,2}^5 K_{1,3}^5}{K_{2,3}^5 - f_7(K^5; x)} \quad \text{by (27),(28)}$$

$$(90) \quad = \frac{K_{0,2}^5 K_{1,3}^5}{f_7(K^4; q) - f_7(K^5; x)} \quad \text{by (82)}$$

$$(91) \quad = \frac{K_{0,2}^5 K_{1,3}^5}{f_7(K^4; q) - f_7(K^4; x)} \quad \text{by (86)}.$$

Together with (39) and (88), this yields

$$(92) \quad \Sigma(K^5; x_0) = \sigma(K^5; x_0) + f_9(K^5; x_0) = \sigma(K^4; x_0) + \frac{K_{0,2}^5 K_{1,3}^5}{f_7(K^4; q) - f_7(K^4; x_0)}.$$

Therefore, by (84),

$$(93) \quad \Sigma(K^5; x_0) = K_{0,1}^5.$$

By (81) and (86), $f_7(K^5; x)$ is positive on $[x_0, q]$, and thus by (40) it is increasing on this same interval. By (87), then, $f_8(K^5; x)$ is decreasing on $[x_0, q]$ and $f_8(K^5; q) = 0$, so $f_8(K^5; x)$ is positive on $[x_0, q]$. Therefore, by (28),

$$(94) \quad f_9(K^5; x) \text{ is singular at } x = q$$

and $f_9(K^5; x)$ is positive on $[x_0, q)$. In particular, by (40),

$$(95) \quad f_9(K^5; x_0) < f_9(K^5; x_1).$$

Together with (81) and (86), the previous paragraph shows that $[x_0, q) \subseteq \chi(K^5)$. Thus, in order to show that there are 3 distinct conductivities on X with response matrix K^5 , it suffices to show that there are 3 distinct values of c in $[x_0, q)$ with $\Sigma(K^5; c) = K_{0,1}^5$. By (93), x_0 is one such c . On the other hand, by (76) and (88) we have

$$(96) \quad \sigma(K^5; x_0) = \sigma(K^4; x_0) = \sigma(K^4; x_1) = \sigma(K^5; x_1).$$

Therefore, we have

$$\begin{aligned} \Sigma(K^5; x_0) &= \sigma(K^5; x_0) + f_9(K^5; x_0) && \text{by (39)} \\ &= \sigma(K^5; x_1) + f_9(K^5; x_0) && \text{by (96)} \\ &< \sigma(K^5; x_1) + f_9(K^5; x_1) && \text{by (95)} \\ &= \Sigma(K^5; x_1) && \text{by (39)}. \end{aligned}$$

Next, we observe that

$$(97) \quad f_9(K^5; x) = \frac{r}{2(p-x_0)} \cdot \frac{1}{\partial_x f_7(K^4; p)} \cdot \frac{(f_7(K^4; q) - f_7(K^4; p))^2}{f_7(K^4; q) - f_7(K^4; x)}$$

by (83) and (91). By the Chain Rule, we therefore have

$$(98) \quad \partial_x f_9(K^5; p) = \frac{r}{2(p-x_0)} \cdot \frac{\partial_x f_7(K^4; p)}{\partial_x f_7(K^4; p)} \cdot \frac{(f_7(K^4; q) - f_7(K^4; p))^2}{(f_7(K^4; q) - f_7(K^4; p))^2} = \frac{r}{2(p-x_0)}.$$

By (94), $|\partial_x f_9(K^5; x)|$ is just a constant times $(x-q)^{-2}$. In particular, as $x_0 < p < q$ by definition, (98) shows that

$$(99) \quad |\partial_x f_9(K^5; x)| \leq \frac{r}{2(p-x_0)} \text{ on } [x_0, p].$$

Thus,

$$(100) \quad f_9(K^5; p) < f_9(K^5; x_0) + r.$$

We then obtain

$$\begin{aligned} \Sigma(K^5; p) &= \sigma(K^5; p) + f_9(K^5; p) && \text{by (39)} \\ &= \sigma(K^4; p) + f_9(K^5; p) && \text{by (88)} \\ &= \sigma(K^4; x_1) - r + f_9(K^5; p) && \text{by (80)} \\ &= \sigma(K^4; x_0) - r + f_9(K^5; p) && \text{by (76)} \\ &= \sigma(K^5; x_0) - r + f_9(K^5; p) && \text{by (88)} \\ &< \sigma(K^5; x_0) + f_9(K^5; x_0) && \text{by (100)} \\ &= \Sigma(K^5; x_0) && \text{by (39)}. \end{aligned}$$

As $\Sigma(K^5; x_1) > \Sigma(K^5; x_0)$ was shown above, the Intermediate Value Theorem implies that there is some $c \in (x_1, p)$ with $\Sigma(K^5; c) = K_{0,1}^5$ (where we are of course using (93)). Finally, since $f_9(K^5; x) \rightarrow +\infty$ as $x \rightarrow q^-$ by (40) and (94), while $\sigma(K^5; x)$ is positive (hence continuous, hence bounded) on $[x_0, q]$ by (38), (81), and (88), it follows from the Intermediate Value Theorem and the inequality $\Sigma(K^5; p) < \Sigma(K^5; x_0)$ shown above that there is a value of c in (p, q) with $\Sigma(K^5; c) = K_{0,1}^5$ (again, given (93)). As $f_9(K^5; x)$ and $f_{18}(K^5; x)$ both have singularities by (77), (86), and (94), our comments at the outset imply that there are precisely three distinct conductivities on X with response matrix K^5 .

7. GENERALIZATION TO ARBITRARY n

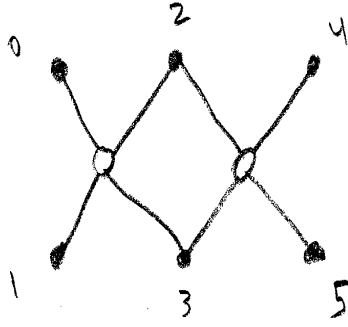
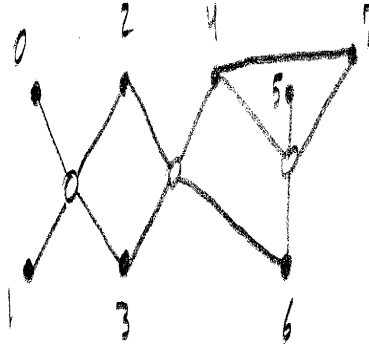
In this section we sketch an extension of the above argument to show that n -to-1 graphs exist for each $n > 1$. We begin with the case $n > 3$; the case $n = 2$ is handled by other methods (discussed below). For $n > 3$, the idea is to proceed inductively, finding a class of graphs $\{X_n\}_{n=3}^\infty$ such that the argument showing that X_n is n -to-1 for some n ‘carries over’ with only slight additions/modifications to show that X_{n+1} is $(n+1)$ -to-1.

We begin by defining a precise notion of ‘gluing together’ finitely many graphs in $\underline{\underline{G}}$, and observing that this operation is well-behaved with respect to star- K transformations. The hypotheses for the definition are as follows: suppose $P_0, \dots, P_n \in \underline{\underline{G}}$ are given and \sim is an equivalence relation on $\coprod_0^n \partial P_i$ such that

- (1) for each $i \neq j$ there exist $m \geq 0$ and integers $0 \leq k_l \leq n$ (for $0 \leq l \leq m$) such that $k_0 = i$, $k_m = j$, and for all $0 \leq l < m$ there exists $a_l \in \partial P_{k_l}$ and $b_l \in \partial P_{k_{l+1}}$ such that $a_l \sim b_l$, and
- (2) if a and b are distinct boundary nodes in some P_i then $a \not\sim b$.

Define \sim on $\coprod_0^n P_i^o$ to be the trivial equivalence relation $a \sim b$ iff $a = b$. (This is done mostly for notational uniformity below.) We define a graph $\bigwedge_0^n P_i$ (which depends on \sim) as follows:

- (1) the boundary of $\bigwedge_0^n P_i$ is the set $\coprod_0^n \partial P_i / \sim$,
- (2) the interior of $\bigwedge_0^n P_i$ is the set $\coprod_0^n P_i^o / \sim$,
- (3) if a and b are nodes in $\bigwedge_0^n P_i$, then for each i , for each $c, d \in P_i$ with $[c] = a$ and $[d] = b$, and for each edge in P_i joining c and d , there is an edge in $\bigwedge_0^n P_i$ joining a and b ,


 FIGURE 8. The graph A .

 FIGURE 9. The graph B .

- (4) the numbering of the nodes in $\partial \bigwedge_0^n P_i$ satisfies the following rule: if $a \in \partial P_i$, $b \in \partial P_j$, and there do not exist c and k such that $c \in P_k$, $[b] = [c]$, and either $k < i$ or $k = i$ and $c < a$ in P_i , then $[a] \leq [b]$, and
- (5) the numbering of the nodes in $(\bigwedge_0^n P_i)^\circ$ satisfies the following rule: if $a, b \in P_i^\circ$ and $a \leq b$ then $[a] \leq [b]$, and if $a \in P_i^\circ, b \in P_j^\circ$, and $i < j$, then $[a] < [b]$.

Unraveling the definitions, one easily proves the following:

Lemma 7.1. *If the P_i and \sim are as above, then $\bigwedge_0^n P_i$ is well-defined, is an element of $\underline{\mathcal{G}}$, and satisfies $(\bigwedge_0^n P_i)^* = \bigwedge_0^n P_i^*$.*

Let S_{n+1} denote the $(n+1)$ -star, with $n+1$ boundary nodes, each adjacent to a single interior node. Let A denote the graph in Figure 8, and let B denote the graph in Figure 9.

We now define the graphs X_n . Fix $n \geq 3$. For $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, let $P_i^n = A$, and for $\lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n$, let $P_i^n = B$. Let $P_0^n = S_{n+1}$. Define \sim by identifying all nodes labeled 4 in copies of A with all nodes labeled 7 in copies of B , all nodes labeled 5 in copies of A with all nodes labeled 6 in copies of B , all nodes labeled 0 in copies of A or B with node 0 in S_{n+1} , and the node labeled 1 in P_i^n ($1 \leq i \leq n$) with the node i in S_{n+1} . We set $X_n = \bigwedge_0^n P_i^n$.

The graph X_2 is the graph shown in Figure 1.

One can check that the graph X_3 is slightly different than the 3-to-1 graph X considered above, although this is not really an important point, i.e., the argument given above for the graph X is easily modified to handle the graph X_3 . Moreover, since each X_n ($n \geq 3$) is obtained by attaching copies of A and B to an $(n+1)$ -star (and then identifying a few nodes on the A s and B s) in a very particular way, and in particular as X_{n+1} is basically like X_n with an extra A or B attached to the central star structure (which has had another boundary node added to accommodate this additional A or B), it is not difficult to see how an argument for X_n would extend to an argument for X_{n+1} in an ‘inductive’ manner. Details will be presented in a later paper. The 2-to-1-ness of X_2 has been established by previous REU students, although the method we applied to (the graph X closely related to) X_3 will in some sense ‘degenerate’ to yield an argument for X_2 as well.