Determining the Resistors in a Network

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Abstract

In this paper we show that the resistors in a rectangular network can be determined by measurements at the boundary of the voltages generated by imposed currents. We also give an algorithm for using the boundary measurements to compute the resistances.

Key words. network of resistors, inverse problem, conductivity.

AMS(MOS) **Subject classification** 31A25, 31B25, 35R30, 90B10.

1 Introduction

We consider a rectangular network of resistors in the plane. Let Z^2 be the lattice in R^2 consisting of the points with integer coordinates. Two lattice points p and q are said to be *adjacent* if there is a horizontal or vertical segment of length one joining them. (The points p and q will also be called *neighbors*, and the line segment joining them will be called pq.) Suppose given integers (a, b) with $a < b$, and (c, d) with $c < d$. A *rectangular network* Ω is constructed as follows. The *nodes* of Ω are the lattice points $p = (i, j)$ for which $a \leq i \leq b$ and $c \leq j \leq d$, with the four corner points $(a, c), (b, c), (a, d)$ and (b, d) excluded. The set of nodes will be denoted Ω_0. For each lattice point p, the set of four adjacent lattice points will be called $\mathcal{N}(p)$. The *interior* int Ω_0 of Ω_0 consists of those nodes p all of whose neighbors are in Ω_0. The *boundary* $\partial\Omega_0$ is Ω_0 - int Ω_0. Every boundary node p has exactly one neighbor in Ω_0, which is an interior node. An *edge* of Ω is the horizontal and vertical line segment $\sigma = pq$ which connects a pair of adjacent nodes p and q in int Ω_0, or which connects a boundary node p to its adjacent interior node q. The set of edges will be denoted Ω_1. Figure 1.1 shows a network with 49 edges, 20 interior nodes and 18 boundary nodes.

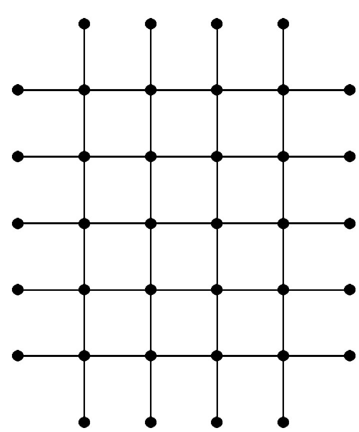

Figure 1.1

A *network of resistors* $\Gamma = (\Omega_0, \Omega_1, \gamma)$ is a network $\Omega = (\Omega_0, \Omega_1)$ together with a function $\gamma : \Omega_1 \rightarrow R^+$ where R^+ is the set of positive real numbers. For each edge $\sigma = pq$ in Ω_1, the number $\gamma(\sigma)$ is called the *conductance* of σ, and

$1/\gamma(\sigma)$ is the *resistance* of σ. The function γ on Ω_1 is called the *conductivity.* For any function $f : \Omega_0 \to R$, we define a function $L_\gamma f : \text{int } \Omega_0 \to R$ by

$$L_\gamma f(p) = \sum_{q \in \mathcal{N}(p)} \gamma(pq)(f(q) - f(p))$$

A function $f : \Omega_0 \to R$ which satisfies $L_\gamma f(p) = 0$ for all interior nodes will be called γ-*harmonic.* If voltage $\phi(r)$ is applied at each boundary node r, the network Ω will acquire a unique voltage $f(p)$ at every interior node p according to *Kirkhoff's Law*, which states that for each interior node p, $L_\gamma f(p) = 0$. (See Section 2.) The function ϕ defined on the boundary nodes determines a current $I_\phi(r)$ through each boundary node r, by $I_\phi(r) = \gamma(rq)(f(r) - f(q))$, where q is the unique neighbor of r in int Ω_0. For each conductivity γ, a quadratic form Q_γ is defined as follows. For two boundary functions ψ and ϕ,

$$Q_\gamma(\psi, \phi) = \sum_{r \in \partial\Omega_0} \psi(r) I_\phi(r)$$

The main result of this paper (Theorem 3.2) is the solution of the inverse conductivity problem for a network of resistors: γ is uniquely determined by the quadratic form Q_γ. Suppose that $\Omega = (\Omega_0, \Omega_1)$ is a network with m boundary nodes. Let F be the space of quadratic forms on R^m. Let T be the map from from conductivities on Ω_1 to F defined by $T(\gamma) = Q_\gamma(\, , \,)$. We calculate the differential dT and show that T is an embedding of the space of conductivities onto a submanifold of F.

Remark 1 Our approach gives a direct method for calculating the conductivity of each resistor in the network.

Remark 2 For the sake of clarity in this paper we have given the proofs for networks in the plane. Similar methods apply to all dimensions higher than two.

Remark 3 The restriction to *rectangular* networks is also unnecessary. The main results (*e.g.* Theorems 3.2 and 5.1) are true for general finite subnetworks of the integer lattice of Z^n. (See the end of Section 3.)

Remark 4 The argument for the calculation of the differential dT follows the pattern originally given by A. Calderon [3].

Remark 5 We have benefitted from discussions with G. Uhlmann and J. Sylvester, who introduced us to inverse problems associated with the conductivity equation $Div(\gamma Grad(u)) = 0$. For the conductivity equation (in dimension at least three), they proved the uniqueness in [6]. The continuity of the inverse was shown by Allessandrini in [2].

2 Preliminaries

In this section, we establish some facts about γ-harmonic functions.

Lemma 2.1 *Let f be a γ-harmonic function on Ω, and let p be an interior node. Then either $f(p) = f(q)$ for all nodes $q \in \mathcal{N}(p)$ or else there is at least one node $q_1 \in \mathcal{N}(p)$ for which $f(p) > f(q_1)$ and there is at least one node $q_2 \in \mathcal{N}(p)$ for which $f(p) < f(q_2)$.*

Proof: By 1.1,

$$\{ \sum_{q \in \mathcal{N}(p)} \gamma(pq) \} f(p) = \sum_{q \in \mathcal{N}(p)} \gamma(pq) f(q)$$

That is, $f(p)$ is the weighted average of its neighbors, with positive weights. If the value of f at some neighbor is less than $f(p)$, then the value at some other neighbor is greater than $f(p)$. QED

Corollary 2.2 *Let f be a γ-harmonic function on Ω_0. Then the maximum and minimum values of f occur on the boundary of Ω_0.*

Proof: Suppose that the maximum value occurs at $p_0 \in$ int Ω_0, and that $f(p_0) > f(q)$ for every $q \in \partial\Omega_0$. Let $\{p_0, p_1, \ldots, p_n\}$ be a sequence of nodes in Ω_0 such that each $p_j p_{j+1}$ is an edge in Ω_1 and $p_n \in \partial\Omega_0$. Then let j be the first index for which $f(p_j) < f(p_0)$. Then $f(p_{j-1}) = f(p_0) \geq f(q)$ for all $q \in \mathcal{N}(p_{j-1})$ and $f(p_{j-1}) > f(p_j)$. This would contradict Lemma 2.1. Similarly for the minimum. QED.

Corollary 2.3 *Let $f : \Omega_0 \to R$, be a function such that $L_\gamma f(p) = 0$ for all $p \in$ int Ω_0, and $f(p) = 0$ for all $p \in \partial\Omega_0$. Then $f(p) = 0$ for all p.*

Proof: Immediate from Corollary 2.2.

Proposition 2.4 *Let functions $h : int\ \Omega_0 \to R$, and $\phi : \partial\Omega_0 \to R$ be given. Then there is a unique function $f : \Omega_0 \to R$ such that $L_\gamma f(p) = h(p)$ for all $p \in int\ \Omega_0$, and $f(p) = \phi(p)$ for all $p \in \partial\Omega_0$.*

Proof: Consider the square system of linear equations for unknowns $f(p)$:

$$L_\gamma f(p) = h(p) \quad \text{for } p \in int\ \Omega_0$$

$$f(p) = \phi(p) \quad \text{for } p \in int\ \partial\Omega_0$$

This system of equations has a unique solution since

$$L_\gamma f(p) = 0 \quad \text{for } p \in int\ \Omega_0$$

$$f(p) = 0 \quad \text{for } p \in int\ \partial\Omega_0$$

has zero as its unique solution, by Corollary 2.3.

We need a discrete version of Green's Formula. Let $(p_0, p_1, \ldots, p_n)$ be the nodes along a horizontal (or vertical) line in Ω_1 as in Figure 2.1.

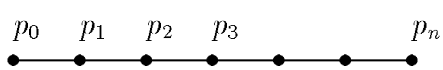

Figure 2.1

Lemma 2.5 *Let V_0 be the collection of nodes p_i, and let E_1 be the collection of edges $p_i p_{i+1}$. Let f and g be two functions $f, g : V_0 \to R$ and let γ be a function $\gamma : E_1 \to R$. Let $f_i = f(p_i)$, $g_i = g(p_i)$, and $\gamma_i = \gamma(p_i p_{i+1})$. Then*

$$\begin{aligned}
\sum_{i=0}^{n-1} \gamma_i (f_{i+1} - f_i)(g_{i+1} - g_i) &= \gamma_0 g_0 (f_0 - f_1) \\
&\quad - g_1[\gamma_0(f_0 - f_1) + \gamma_1(f_2 - f_1)] \\
&\quad - g_2[\gamma_1(f_1 - f_2) + \gamma_2(f_3 - f_2)] \\
&\quad + \cdots + \gamma_{n-1} g_n (f_n - f_{n-1})
\end{aligned}$$

Proof: Rearrange the terms. QED

Notation Henceforth, let the nodes in Ω_0 be indexed p_i. For a function $f : \Omega_0 \to R$, let $f_i = f(p_i)$. For a function $\gamma : \Omega_1 \to R$, let $\gamma_{ij} = \gamma(\sigma_{ij})$ for each edge $\sigma_{ij} = p_i p_j \in \Omega_1$. For each boundary node p_i, let e_i be the unique edge connecting it to its adjacent interior node denoted $nb(p_i)$.

Proposition 2.6 (Discrete Green's Formula) *Let $\Omega = (\Omega_0, \Omega_1)$ be a network, and let $\gamma : \Omega_1 \to R$ and $f, g : \Omega_0 \to R$ be functions. Then*

$$\begin{aligned} \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) &= \sum_{p_i \in \partial\Omega_0} g_i \gamma(e_i)(f(p_i) - f(nb(p_i))) \\ &\quad - \sum_{p_i \in int\,\Omega_0} g_i L_\gamma f(p_i) \end{aligned}$$

Proof: Use Lemma 2.5 and compute the sum on the left by first summing horizontally and then vertically.

Corollary 2.7 *If $\Gamma = (\Omega_0, \Omega_1, \gamma)$ is a network of resistors and f and g are γ-harmonic functions on Ω_0 with $g = \psi$ and $f = \phi$ on $\partial\Omega_0$, then*

$$\begin{aligned} Q_\gamma(\psi, \phi) &= \sum_{p \in \partial\Omega_0} \psi(p) I_\phi(p) \\ &= \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)(g_i - g_j) \\ &= \sum_{p \in \partial\Omega_0} \phi(p) I_\psi(p) \\ &= Q_\gamma(\phi, \psi) \end{aligned}$$

where the above notation is used.

Proof: Immediate from Proposition 2.6, using $L_\gamma(f) = 0$ and $L_\gamma(g) = 0$. QED

We will use a process which we call harmonic continuation. Let $\Gamma = (\Omega_0, \Omega_1, \gamma)$ be a network of resistors, and let the columns of Ω_0 be $C_0, C_1, \ldots, C_n$, numbering from left to right. Let S be the subset of Ω_0 consisting of the nodes in columns $C_0, C_1, \ldots, C_k$. Suppose that f is is a function defined on S which is γ-harmonic on the nodes which are interior in S.

Lemma 2.8 *In this situation f can be defined on C_{k+1}, so that f is harmonic on the larger set. The definition of f is uniquely determined on the interior nodes of C_{k+1}, and can be given arbitrary values on the endnodes of column C_{k+1}.*

Proof: First notice that the definition of f on the nodes of column C_{k+1} will not affect the assumed harmonicity of f at any of the nodes in columns $C_0, C_1, \ldots, C_{k-1}$. We consider a node p in column C_k, with its four neighbors q_i , as in Figure 2.2.

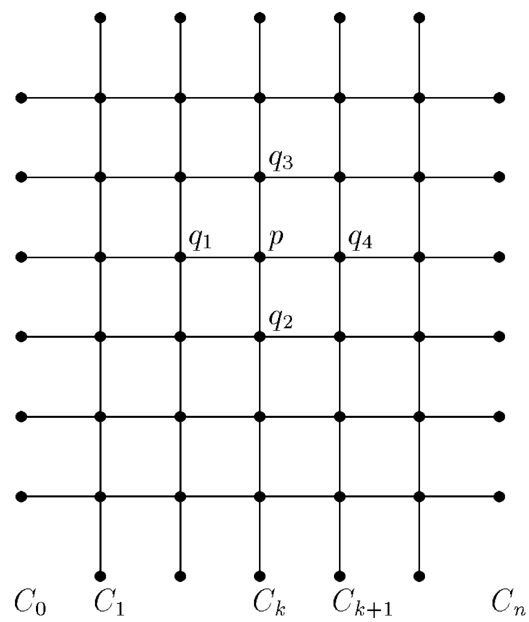

Figure 2.2

Kirkhoff's Law implies that we must have

$$\{\sum_{q\in N(p)} \gamma(pq)\} f(p) = \sum_{q\in N(p)} \gamma(pq) f(q)$$

If q is an interior node of C_{k+1}, then f(q) is determined by the values to the left. In Figure 2.2, $f(q_4)$ is determined by $f(p)$, $f(q_1)$, $f(q_2)$ and $f(q_3)$. The values f on the two boundary nodes of C_{k+1} can be assigned arbitrarily. QED

Harmonic continuation is also valid to the left, up or down. Note that although the values of f on the boundary nodes of C_k are arbitrary, these values affect the next step of the continuation. Consider Figure 2.3, where the dotted line is the diagonal of slope -1 passing through the top node of C_k.

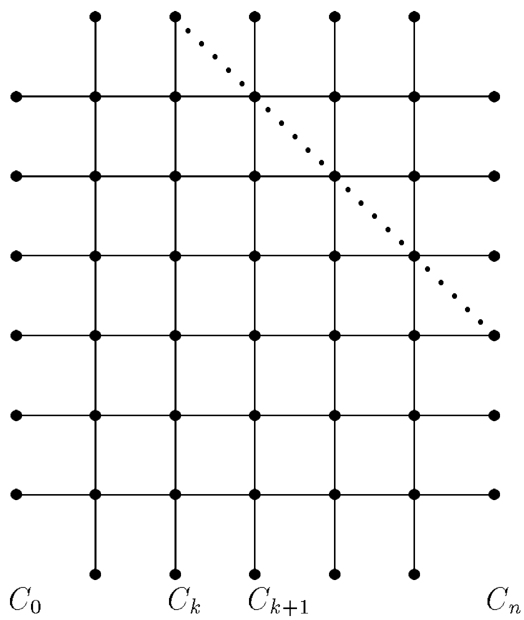

Figure 2.3

Lemma 2.9 *Let $\Gamma = (\Omega_0, \Omega_1, \gamma)$ be a network of resistors. Suppose a function f is defined and constant on the nodes of columns $C_0, C_1, \ldots, C_k$. Then f can be continued as a γ-harmonic function where f is constant on or below the diagonal indicated by the dotted line. The values of f at boundary nodes at the tops of columns $C_{k+1}, \ldots C_n$ are arbitrary.*

Proof: Immediate from Lemma 2.8. QED

This shows that it is possible for a γ-harmonic function to be locally constant, without being constant throughout Ω. This is in contrast to the continuous case, where a harmonic function which is constant on an open set must be identically constant.

We need some facts (well-known) about the discrete Neumann problem. Let f be a real function defined on Ω_0 with $L_\gamma f = 0$. Let $\phi = f|_{\partial\Omega_0}$ and

$I_\phi(p) = \gamma(pq)[f(p) - f(nb(p))]$ where $nb(p)$ is the unique neighbor of p in int Ω_0. Then we have the following.

Lemma 2.10 *Suppose $L_\gamma f(p) = 0$ for $p \in int\ \Omega_0$ and $I_\phi(r) = 0$ for $r \in \partial\Omega_0$. Then there is a $c \in R$ in such that $f(p) = c$ for all $p \in \Omega_0$.*

Proof: Since $L_\gamma f = 0$,

$$Q_\gamma(\phi, \phi) = \sum_{\sigma_{ij} \in \Omega_1} \gamma_{ij}(f_i - f_j)^2 = \sum_{p \in \partial\Omega_0} \phi(p) I_\phi(p) \tag{1}$$

Since $I_\phi(p) = 0$ for $p \in \partial\Omega_0$, $Q_\gamma(\phi, \phi) = 0$. Thus $f(p_i) = f(p_j)$ for all $p_i, p_j \in \Omega_0$. QED

Suppose values $\{J_p\}$ are given; consider the system of equations (one equation for each node in Ω_0):

$$\begin{aligned} L_\gamma f(p) &= 0, \text{ for } p \in int\, \Omega_0 \\ \gamma(pq)[f(q) - f(p)] &= -J_p, \text{ for } p \in \partial\Omega, q = nb(p) \end{aligned} \tag{2}$$

The unknowns are the values of f.

Lemma 2.11 *This system of equations has a solution if and only if*

$$\sum_{p \in \partial\Omega_0} J_p = 0$$

If f and g are two solutions then $f - g$ is constant.

Proof: (This is an application of the Fredholm alternative to the discrete situation). Let B_γ be the matrix of the system of equations (3) and let $e = [1, 1, \ldots, 1] \in R^n$, where n is the number of nodes. Then it is easy to see that

$$e \cdot B_\gamma = 0 \tag{3}$$

By Lemma 2.10, $\ker(B_\gamma) = \{c \cdot e^T : c \in R\}$. By (4) the range of B_γ is orthogonal to the kernel of B_γ (B_γ is a square matrix). Since $\dim(\ker(B_\gamma)) = 1$, B_γ maps onto $\{b : e \cdot b = 0\}$. Since $\sum_{p \in \partial\Omega_0} J_p = 0$, there is a solution of (3). By Lemma 2.10, it is unique up to a constant. QED

3 Global uniqueness

Let $\Omega = (\Omega_0, \Omega_1)$ be a rectangular network with N edges and m boundary nodes. We show that the map T from $(R^+)^N$ to the space F of quadratic forms on R^m is $1-1$. Observe that

$$\begin{aligned} \sum_{p_j \in \partial\Omega_0} \phi(p_j) I_\psi(p_j) &= Q_\gamma(\phi, \psi) \\ &= \frac{1}{2}[Q_\gamma(\phi+\psi, \phi+\psi) - Q_\gamma(\phi,\phi) - Q_\gamma(\psi,\psi)] , \end{aligned} \quad (4)$$

Thus knowing $Q_\gamma(\phi, \phi)$ for all ϕ is equivalent to knowing the *Dirichlet to Neumann map* Λ_γ, (or its inverse the *Neumann to Dirichlet map*). Λ_γ maps the boundary value function $\phi(p)$, $p \in \partial\Omega_0$ to the current function $I_\phi(p)$, $p \in \partial\Omega_0$ which is determined by the solution to the Dirichlet problem with boundary values ϕ. (The boundary values and boundary currents are dually paired by equation (5)). In this sense we show that measurements at the boundary determine γ.

At each corner of the rectangular network there are two edges, each containing a boundary node. We first show how to determine the conductances of these edges.

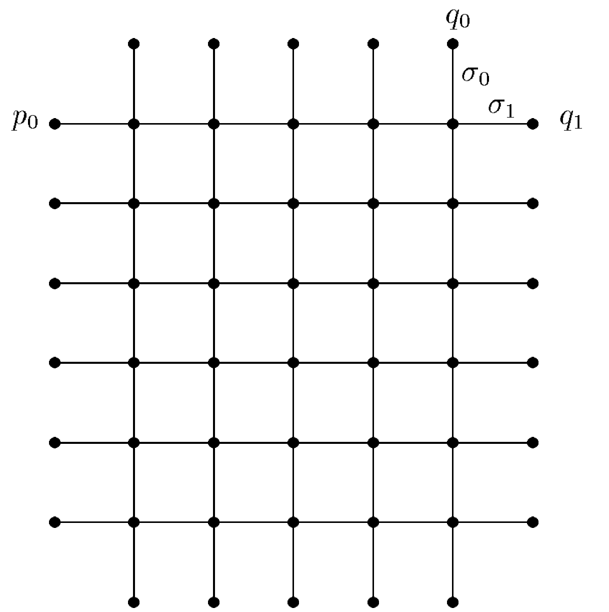

Figure 3.1

Referring to Figure 3.1, we want to compute the conductances $\gamma(\sigma_0)$ and $\gamma(\sigma_1)$. Consider the following Neumann problem. The current is set equal to 0 at all boundary nodes except at the corner pair, where the current is 1 at node q_0 and -1 at node q_1. To uniquely determine the solution, the voltage is set equal to 0 at the boundary node p_0 at the top left. By Lemma 2.9 we know that there is a (unique) γ-harmonic function with values zero everywhere except in the upper right corner. This is also the solution of the Neumann problem we have just posed which is unique by Lemma 2.11. Thus by measuring the voltages $f(q_0)$ and $f(q_1)$, we know the conductances $\gamma(\sigma_0) = f(q_0)^{-1}$ and $\gamma(\sigma_1) = -f(q_1)^{-1}$ of these edges. Referring to Figure 3.2, consider the conductances of each edge within the strip bounded by the diagonal lines (dotted) of slope -1.

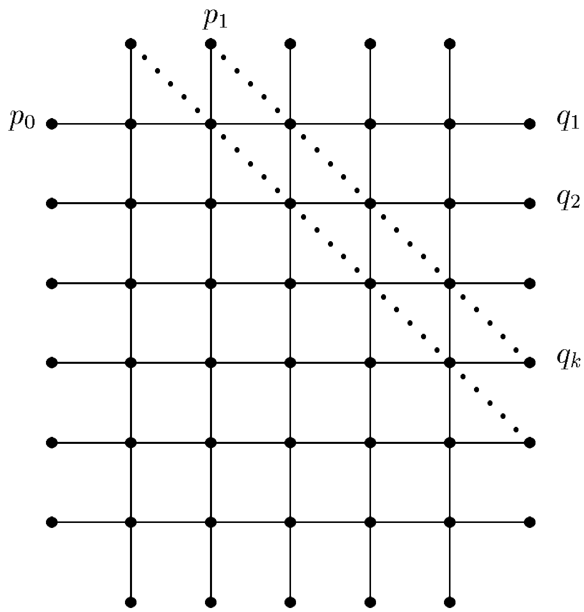

Figure 3.2

Assume inductively that we know the conductances of each edge above the diagonal from p_1 to q_k. By Lemma 2.9 there is a γ-harmonic function h such that h is 0 on and below the lower diagonal, has current 1 at the node p_1, and has current 0 at all other exterior nodes except at nodes q_j for $j = 1, \ldots, k$. Suppose the current at q_j is $-\alpha_j$.

Lemma 3.1 *The numbers $\alpha_1, \ldots, \alpha_k$ are uniquely determined by the conditions on the currents at the other exterior nodes and the condition that h have the same value at all boundary nodes on the left side.*

Proof: This follows from Lemma 2.9. QED

For each $j = 1, 2, \ldots, k$, let h_j be the solution of the Neumann problem:

$$\begin{aligned} I_{h_j}(q) &= 0, \, if \; q \neq p_1, q_j \\ I_{h_j}(p_1) &= 1 \\ I_{h_j}(q_j) &= -1 \\ h_j(p_0) &= 0 \end{aligned} \tag{5}$$

Then $\sum_{j=1}^{k} \alpha_j h_j$ solves the same Neumann problem as h and thus

$$\sum_{j=1}^{k} \alpha_j h_j = h \tag{6}$$

Now we find $\{\alpha_1, \ldots, \alpha_k\}$. We know that (6) has a unique solution. We also know that if

$$\sum_{j=1}^{k} \alpha_j = 1$$

$$\sum_{j=1}^{k} \alpha_j h_j(q) = 0 \tag{7}$$

for all q which are boundary nodes of the left column, then (7) holds by Lemma 3.1. Hence the (sometimes) overdetermined system (8) has a unique solution $\{\alpha_1, \ldots, \alpha_k\}$. Thus by using the solutions of the Neumann problems (6) and by solving (8) we can find the function h. We now have a γ-harmonic function h with known values and currents at all boundary nodes. The values of h are in fact 0 at all nodes on and to the left of the lower of the two dotted diagonals. Moreover the values of h are also known at all neighbors of boundary nodes since these values are either known to be zero or can be computed from known conductances, currents, and boundary values. By using Kirkhoff's Law and known conductances we find the values of h at all remaining nodes. We will use the function h and the inductively known conductances above the diagonal to compute the conductances within the diagonal strip.

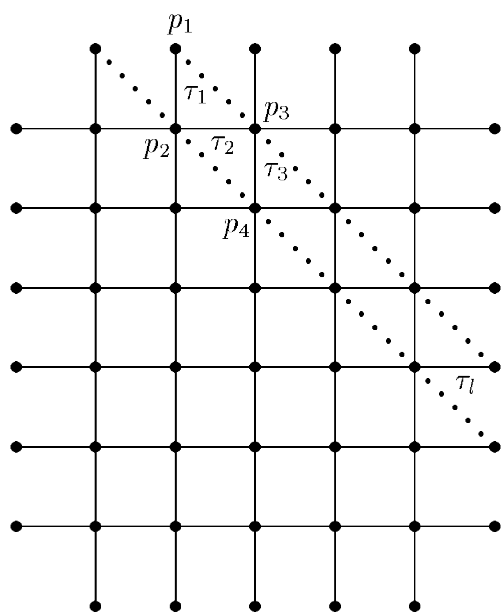

Figure 3.3

In Figure (3.3) the letters τ_j stand for the conductances of resistors in the diagonal strip. The letters p_j stand for the nodes in this strip. We first compute the conductance τ_1 from the known current through from p_1 to p_2 (it is 1) and the voltage drop from p_1 to p_2 (it is the value of h at p_1). We next use Kirkhoff's law at p_2 to compute τ_2. (Three currents are known and the voltages at p_2 and p_3 are known.) We can then compute τ_3 by using Kirkhoff's law at p_3 and the known voltages and conductances. We continue until we finally compute τ_l ($l = 2k$). By going to the right, left, up or down, we can inductively find the conductance of each resistor in the network. We have thus proved the following.

Theorem 3.2 *T is $1-1$. That is, if $\gamma_1 \neq \gamma_2$ then $Q_{\gamma_1} \neq Q_{\gamma_2}$.*

The proof of Theorem 3.2 describes a direct algorithm for calculating γ from the Neumann to Dirichlet map. There is a similar direct algorithm for calculating γ from the Dirichlet to Neumann map.

We now give a justification of Remark 3 of Section 1. Let $\Omega = (\Omega_0, \Omega_1)$ be a general network of the following form. Ω_1 is a finite set of edges pq where

p, q are points in the integer lattice Z^2 and $|p-q| = 1$; $\Omega_0 = \{p \in Z^2 : pq \in \Omega_1$ for some $q\}$. We take the same definition of interior and boundary: int $\Omega_0 = \{p \in \Omega_0 : \mathcal{N}(p) \subset \Omega_0\}$; $\partial\Omega_0 = \Omega_0 -$ int Ω_0. A node $p \in \partial\Omega_0$ may have one, two, or three neighbors in Ω_0. As before, a conductivity function is a function γ from Ω_1 to R^+, and a resistor network is a triple $(\Omega_0, \Omega_1, \gamma)$. In this context if ϕ is a boundary potential, and f is the harmonic function with boundary values ϕ, the current at a boundary node $p \in \partial\Omega_0$ is now be defined as

$$I_\phi(p) = \sum_{q \in \mathcal{N}(p) \cap \Omega_0} \gamma(pq)[f(p) - f(q)]$$

It is straightforward to verify most of the previous results for such a network, but for the proof of Theorem 3.2 in this general setting, we need Proposition 3.3 below.

Let $\Gamma = (\Omega_0, \Omega_1, \gamma)$ be a general resistor network, as above, with conductivity function γ. Adjoin edges to Ω to produce a rectangular network. Make this enlarged network into a resistor network (W_0, W_1, μ) by setting μ equal to 1 on the adjoined edges.

Propostion 3.3 *Λ_γ determines Λ_μ.*

Proof: Let ϕ be given on ∂W_0. We show how to compute $\Lambda_\mu\phi$. Let $\{p_1, \ldots, p_k\}$ be the boundary nodes of Ω_0. If u is a function defined on $\partial\Omega_0$ Λ_γ is regarded as a matrix acting on the vector $[u(p_1), \ldots, u(p_k)]^T$ so that $\Lambda_\gamma u$ is the vector of currents through $\{p_1, \ldots, p_k\}$. Consider the following set of equations.

$$L_\mu f(p) \;=\; 0 \; for \; p \in int\, W_0 - \Omega_0 \qquad (8)$$

$$\Lambda_\gamma[f(p_1), \ldots, f(p_k)]^T \;=\; [J(p_1), \ldots, J(p_k)]^T \; for \; p_j \in \partial\Omega_0 \qquad (9)$$

where $f|\partial W_0 = \phi$. In this set of equations, when $p \in \partial\Omega_0 \cap \text{int}W_0$, we define

$$J(p) = \sum_{q \in \mathcal{N}(p) - \Omega_0} \mu(pq)[f(q) - f(p)] = \sum_{q \in \mathcal{N}(p) - \Omega_0} [f(q) - f(p)]$$

(The second equality holds since $\mu(pq) = 1$ if $pq \in W_1 - \Omega_1$.) The unknown terms in (9) and (10) are the values $f(p)$ for $p \in int\, W_0 - int\, \Omega_0$, and $J(p)$ for

$p \in \partial\Omega_0 \cap \partial W_0$. The number of unknowns equals the number of equations. We now show that this system has a unique solution. We know from Proposition 2.4 that there is a unique solution of the Dirichlet problem:

$$L_\mu h = 0,\ h|\partial W_0 = \phi \tag{10}$$

This function h will be used to find a solution of (9) and (10). For $p \in \partial\Omega_0 \cap \partial W_0$ set $J(p) = \mu(pq)[h(p) - h(nb(p))]$ where $nb(p)$ is the unique neighbor of p in W_0, and set $f(p) = h(p)$ for $p \in int\, W_0 - int\, \Omega_0$. This gives a solution of (9) and (10). This is the only possible solution of (9) and (10), since a different solution of (9) and (10) would lead to a different solution of the the Dirichlet problem (11). Thus we know that the system (9) and (10) has a unique solution. Then Λ_μ is given by

$$\Lambda_\mu \phi(p) \;=\; J(p),\ if\ p \in \partial\Omega_0 \cap \partial W_0 \tag{11}$$

$$\Lambda_\mu \phi(p) \;=\; f(p) - f(nb(p)),\ if\ p \in \partial W_0 - \partial\Omega_0 \tag{12}$$

where f and J are the solutions of (9) and (10). QED

Corollary 3.4 *For a general network of resistors, the conductivity γ is uniquely determined by Λ_γ.*

Proof: This follows from from Proposition 3.3 and Theorem 3.2. QED

4 The differential of T

For the computation of the differential we consider $T(\gamma + \kappa)$ for a small perturbation κ. For any function $\alpha : \Omega_1 \to R$, the *norm* of α is $\|\alpha\| = \max |\alpha(\sigma)|$ for $\sigma \in \Omega_1$. Fix a conductivity γ, and consider a perturbation $\gamma + \kappa$ where $\|\kappa\|$ is small. Consider the solutions of

$$L_{\gamma+\kappa} u(p) = 0 \tag{13}$$

for values $u(p)$ for $p \in \text{int } \Omega_0$, and where the values of $u(p) = \phi(p)$ for $p \in \partial\Omega_0$. Let $u = f + g$, where $f = \phi$ on $\partial\Omega_0$ and $L_\gamma f = 0$ in int Ω_0. Then $g = 0$ on $\partial\Omega_0$, and (14) implies that:

$$(L_\gamma + L_\kappa) g = -L_\kappa f \tag{14}$$

in int Ω_0. If h is given, $L_\gamma^{-1}h$ is defined to be the solution v of $L_\gamma v = h$ with $v = 0$ on $\partial\Omega_0$. By Proposition 2.4, this makes sense. In the present context, this implies that

$$L_\gamma^{-1}L_\gamma g = g$$

since $g = 0$ on $\partial\Omega_0$. Equation (15) yields

$$(I + L_\gamma^{-1}L_\kappa)g = -L_\gamma^{-1}L_\kappa f$$

in int Ω_0. $L_\kappa f$ is linear in κ at all nodes in int Ω_0. Hence $-L_\gamma^{-1}L_\kappa f$ is linear in κ. Moreover if $\|\kappa\|$ is small, then $I + L_\gamma^{-1}L_\kappa$ is invertible on those g with $g(p) = 0$ on $\partial\Omega_0$. Also

$$g = -(I + L_\gamma^{-1}L_\kappa)^{-1}L_\gamma^{-1}L_\kappa f = O(\kappa) \qquad (15)$$

That is, g vanishes to order 1 in κ. We have

$$\begin{aligned}
Q_{\gamma+\kappa}(\phi,\phi) &= \sum_{\sigma_{ij}\in\Omega_1}(\gamma_{ij}+\kappa_{ij})(f_i - f_j + g_i - g_j)^2 \\
&= Q_\gamma(\phi,\phi) + 2\sum_{\sigma_{ij}\in\Omega_1}\gamma_{ij}(f_i - f_j)(g_i - g_j) + \\
&\quad 2\sum_{\sigma_{ij}\in\Omega_1}\kappa_{ij}(f_i - f_j)(g_i - g_j) + \sum_{\sigma_{ij}\in\Omega_1}\gamma_{ij}(g_i - g_j)^2 + \\
&\quad \sum_{\sigma_{ij}\in\Omega_1}\kappa_{ij}(f_i - f_j)^2 + \sum_{\sigma_{ij}\in\Omega_1}\kappa_{ij}(g_i - g_j)^2
\end{aligned}$$

By Proposition 2.6, since $L_\gamma f = 0$, and $g = 0$ on $\partial\Omega_0$, we have

$$\sum_{\sigma_{ij}\in\Omega_1}\gamma_{ij}(f_i - f_j)(g_i - g_j) = 0$$

By equation (16)

$$2\sum_{\sigma_{ij}\in\Omega_1}\gamma_{ij}(f_i - f_j)(g_i - g_j) + \sum_{\sigma_{ij}\in\Omega_1}\gamma_{ij}(g_i - g_j)^2 + \sum_{\sigma_{ij}\in\Omega_1}\kappa_{ij}(g_i - g_j)^2$$

$$= O(\kappa^2)$$

Thus,

$$Q_{\gamma+\kappa}(\phi,\phi) = Q_\gamma(\phi,\phi) + \sum_{\sigma_{ij}\in\Omega_1} \kappa_{ij}(f_i - f_j)^2 + O(\kappa^2) \tag{16}$$

This proves the following. Recall that the space of quadratic forms on R^m is denoted F.

Theorem 4.1 *The map $T : (R^+)^N \to F$ is differentiable. The differential at $\gamma \in (R^+)^N$ as a linear function of $\kappa \in R^N$ is given by the quadratic form*

$$dT_\gamma\kappa(\phi,\psi) = \sum_{\sigma_{ij}\in\Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j)$$

where f and g are γ-harmonic functions on Ω_0, with $f|_{\partial\Omega_0} = \phi$, and $g|_{\partial\Omega_0} = \psi$.

Theorem 4.2 *Let $\Gamma = (\Omega_0, \Omega_1, \gamma)$ be a network of resistors. Suppose given any function $\kappa : \Omega_1 \to R$ and suppose that for all γ-harmonic f and g functions*

$$\sum_{\sigma_{ij}\in\Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) = 0$$

Then $\kappa(\sigma) = 0$ for all $\sigma \in \Omega_1$. Hence the differential $dT : R^N \to R^{m^2}$ is one to one.

Proof: We order the edges (from the outside in) as follows. First every node is assigned a *level* by $level(p) = min|p - r|$, where r is a boundary node. The nodes of a fixed level form the sides of a rectangle in Ω_0. In the notation for an edge $\sigma = pq$, it will be assumed that $level(p) \leq level(q)$. Each edge $\sigma = pq$ is assigned a level by $level(pq) = level(p) + level(q)$. The edges are partially ordered by level. By ordering the edges arbitrarily within each level, we obtain a total ordering of the edges. There are two types of edges, those whose level is odd, and those whose level is even.

Let $\sigma = pq \in \Omega_1$, be an edge of level one as indicated in Figure 4.1.

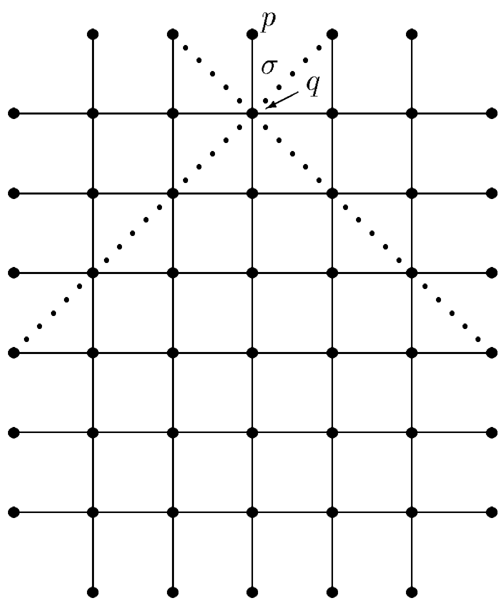

Figure 4.1

First we show that $\kappa(\sigma) = 0$. Let f be a γ-harmonic function with $f(p) = 1$, and $f(r) = 0$ on all nodes r on or below the diagonal of slope -1 which includes q. Similarly, let g be a γ-harmonic function with $f(p) = 1$, and $f(r) = 0$ on all nodes r on or below the diagonal of slope +1 which includes q. For $rs \neq \sigma$, $(f(r) - f(s))(g(r) - g(s)) = 0$. Furthermore, $(f(p) - f(q))(g(p) - g(q)) = 1$. Thus

$$\sum_{\sigma_{ij} \in \Omega_1} \kappa_{ij}(f_i - f_j)(g_i - g_j) = \kappa(\sigma) \tag{17}$$

By hypothesis, the sum is 0, so $\kappa(\sigma) = 0$.

Consider next an edge of level two like $\sigma = pq$ in Figure 4.2.

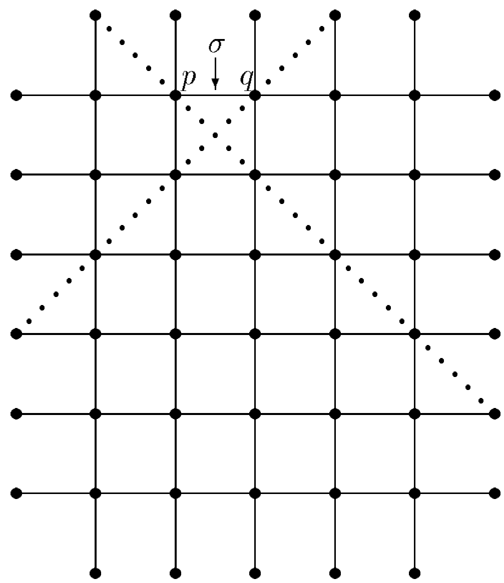

Figure 4.2

By using Lemma 3.1 as in Lemma 3.2, there is a γ-harmonic function f with $f(p) = 0, f(q) = 1$ and $f(r) = 0$ for all nodes r on or below the diagonal of slope 1 which includes p. There is also a γ-harmonic function g with $g(p) = 0, g(q) = 1$ and $g(r) = 1$ for all nodes r on or below the diagonal of slope $+1$ which includes q. We have already shown that $\kappa(\tau) = 0$ for each edge τ of level one. For this choice of f and g, $(f(p) - f(q))(g(p) - g(q)) = 1$. For any other edge rs , either $(f(r) - f(s))(g(r) - g(s)) = 0$ or $\kappa(rs) = 0$. Thus formula (6) again holds. By hypothesis, the sum is 0, so $\kappa(\sigma) = 0$. This argument shows that $\kappa(\sigma) = 0$ for each edge σ of level two. Considering next an edge of level three such as σ in figure 4.3, we construct γ-harmonic functions f, g which show that $\kappa(\sigma) = 0$. Continuing in this way, level by level, we find that $\kappa(\sigma) = 0$ on all edges. QED

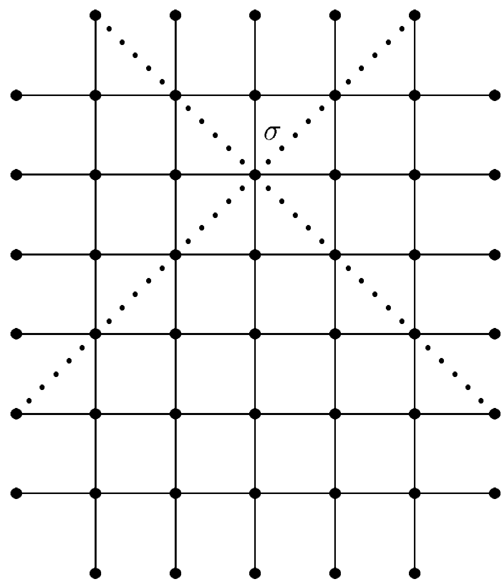

Figure 4.3

For a given conductivity γ on Ω and boundary potential ϕ the *power* dissipated by Ω is $Q_\gamma(\phi,\phi)$. The following proposition proves a kind of monotonicity for the power dissipated.

Proposition 4.3 *Suppose conductances γ^0 and γ^1 satisfy $\gamma^1(\sigma) \geq \gamma^0(\sigma)$ for all $\sigma \in \Omega_1$ and $\gamma^1(\tau) > \gamma^0(\tau)$ for some $\tau \in \Omega_1$. Then there is a ϕ such that $Q_{\gamma^1}(\phi,\phi) > Q_{\gamma^0}(\phi,\phi)$.*

Proof: Let $\gamma^t = (1-t)\gamma^0 + t\gamma^1$, and let ϕ be any boundary function and define $s_\phi(t) = Q_{\gamma^t}(\phi,\phi)$. Then

$$s_\phi(1) - s_\phi(0) = \int_0^1 s'_\phi(t)dt$$

But

$$s'_\phi(t) = \sum_{\sigma_{ij}\in\Omega_1} \gamma'_{ij}(t)(f^t_i - f^t_j)^2$$

where $L_{\gamma^t} f^t = 0$ and $f^t = \phi$ on $\partial\Omega_0$. Since $\gamma'_{ij}(t) = \gamma^1_{ij} - \gamma^0_{ij} \geq 0$, it follows that $s_\phi(1) \geq s_\phi(0)$. If $\gamma^1_{kl} > \gamma^0_{kl}$ for some σ_{kl}, then it follows from Theorem 4.2

that there is a γ^0-harmonic function f^0 such that

$$s'_\phi(0) = \sum_{\sigma_{ij}\in\Omega_1} (\gamma^1_{ij} - \gamma^0_{ij})(f^0_i - f^0_j)^2 \neq 0$$

and hence $s'_\phi(0) > 0$, where ϕ is f^0 restricted to $\partial\Omega_0$. Thus $Q_{\gamma^1}(\phi,\phi) > Q_{\gamma^0}(\phi,\phi)$. QED

5 The continuity of T^{-1}

Let $\Omega = (\Omega_0, \Omega_1)$ be a network with N edges and m boundary nodes.

Theorem 5.1 *The map T has a continuous inverse. The image $T((R^+)^N)$ is an embedded submanifold of F, and T is a diffeomorphism onto its image.*

Proof: Let γ_1 and γ_2 be two conductivity functions on Ω_1. Suppose that Q_{γ_1} is close to Q_{γ_2}. Then Λ_{γ_1} is close to Λ_{γ_2}. This means that if ϕ is given then $\Lambda_{\gamma_1}(\phi)$ is close to $\Lambda_{\gamma_2}(\phi)$, and if I is given with $\sum_{p\in\partial\Omega_0} I(p) = 0$ then $\phi_1 = \Lambda^{-1}_{\gamma_1}(I)$ is close to $\phi_2 = \Lambda^{-1}_{\gamma_2}(I)$. (By $\Lambda^{-1}_\gamma(I)$ we mean the function ϕ defined on $\partial\Omega_0$ such that $I_\phi = I$, and such that $\phi(p_0) = 0$, where p_0 is a fixed node on $\partial\Omega_0$ which is used to uniquely define Λ^{-1}_γ and make it a linear map). Now we consider the algorithm for computing T^{-1}. First consider the Neumann problem:

$$\begin{aligned} I(q_0) &= 1 \\ I(q_1) &= -1 \\ I(q) &= 0 \text{ if } q \neq q_0, q_1 \\ \phi(p_0) &= 0 \end{aligned}$$

where the notation refers to Figure 3.1. Let $\phi_1 = \Lambda^{-1}_{\gamma_1}(I)$ and $\phi_2 = \Lambda^{-1}_{\gamma_2}(I)$. Then ϕ_1 is close to ϕ_2, and so $\phi_1(q_0)$ is close to $\phi_2(q_0)$. But $\phi_1(q_0) = 1/\gamma_1(\sigma_0)$ and $\phi_2(q_0) = 1/\gamma_2(\sigma_0)$, and so $\gamma_1(\sigma_0)$ is close to $\gamma_2(\sigma_0)$. Similarly, $\gamma_1(\sigma_1)$ is close to $\gamma_2(\sigma_1)$, and we see that the corner values of γ_1 and γ_2 are close.

Next we consider the Neumann problems of equation (6), which are used in the inductive computation of the values of γ. We assume already proved that the previously computed values of γ are close. The two different solutions of the Neumann problems found by using Λ_{γ_1} and Λ_{γ_2} are denoted by $h_j^{(1)}$ and $h_j^{(2)}$ and are close by assumption. Hence the coefficients of the equations

$$\sum_{j=1}^{k} \alpha_j^{(1)} = 1$$
$$\sum_{j=1}^{k} \alpha_j^{(1)} h_j^{(1)}(q) = 0 \tag{18}$$

are close to the coefficients of the equations

$$\sum_{j=1}^{k} \alpha_j^{(2)} = 1$$
$$\sum_{j=1}^{k} \alpha_j^{(2)} h_j^{(2)}(q) = 0 \tag{19}$$

Thus the solutions $\{\alpha_j^{(1)} : 1 \leq k\}$ and $\{\alpha_j^{(2)} : 1 \leq k\}$ are close. It then follows that the boundary values of the functions

$$h^{(1)} = \sum_{j=1}^{k} \alpha_j^{(1)} h_j^{(1)}$$

and

$$h^{(2)} = \sum_{j=1}^{k} \alpha_j^{(2)} h_j^{(2)}$$

are close. Proceeding inductively we see that the functions γ_1 and γ_2 are close.

By the results of Section 4, the map dT is injective, so $T((R^+)^N)$ is an immersed submanifold. By what we have just proved T is a homeomorphism onto its image. QED

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