RESEARCH STATEMENT

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1. Introduction and background

I am interested in nonlinear elliptic partial differential equations and their applications to geometry. I have recently been applying tools from the theories of elliptic equations, minimal surface theory, and Lagrangian geometry to obtain a priori estimates for special Lagrangian equations. My recent work and future interest lie in the areas of a priori estimates and calibrated geometry.

1.1. **A priori estimates for fully nonlinear equations.** A priori estimates are the key to the solvability of certain partial differential equations through the continuity method. These estimates also provide the error bound of the numerical approximation to the solutions. For equations with convexity and uniform ellipticity, the theory of a priori estimates is well developed after the work of Krylov and Evans in the early 1980's. However, for many fully nonlinear equations currently of interest, such as the special Lagrangian equations, uniform ellipticity needs to be shown. Thus in order to obtain regularity for these equations, one must begin by proving a priori estimates.

1.2. **Calibrated and Lagrangian Geometry.** On a Riemannian manifold M, a large class of volume minimizing k-submanifolds may be studied using a calibration, a closed exterior k-form on M which is bounded above by the volume form. Harvey and Lawson chartered the subject in 1982 [HL], demonstrating that solutions to (2.1) define volume minimizing submanifolds in $\mathbb{C}^n$. These submanifolds are called *special Lagrangian.* More recently, in 1996, Strominger, Yau and Zaslow [SYZ] suggested that understanding special Lagrangian submanifolds may be the key towards understanding mirror symmetry, a phenomenon discovered by physicists working in string theory in the early 1990's. The SYZ conjecture has strong implications in both mathematics and physics, so has been a focus of recent research in algebraic geometry, differential geometry, and mathematical physics. From the analytic viewpoint, the study of these special Lagrangian submanifolds has seen progress by Rick Schoen and others.

2. Recent results

2.1. **The special Lagrangian equation.** A function u is a solution to the special Lagrangian equation if

$$\sum_{i=1}^{n} \arctan \lambda_i = \Theta \tag{2.1}$$

where $\lambda_i s$ are the eigenvalues of the Hessian $D^2 u\left(x\right)$. The special Lagrangian equation (2.1) arises in calibrated geometry, holding if and only if the gradient graph $\left(x, \nabla u\left(x\right)\right) \subset$

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$\mathbb{C}^n$ is a (volume minimizing) minimal surface in $\mathbb{R}^n \times \mathbb{R}^n$ [HL]. A Lagrangian graph $M = (x, \nabla u(x)) \subset \mathbb{C}^n = \mathbb{R}^n \times \mathbb{R}^n$ is called *special* when it is calibrated and therefore volume minimizing. For $n = 3$ with $\Theta = \pi/2$ or $\Theta = \pi$, the equation (2.1) takes the following forms, respectively

$$\sigma_2(D^2u) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = 1 \tag{2.2}$$

$$\triangle u = \det D^2 u. \tag{2.3}$$

In the past year, Yu Yuan and I have been working on the problem of a priori estimates for special Lagrangian equations. Our results are outlined in the following.

2.1.1. *The equation $\sigma_2 = 1$ for $n = 3$.* In the joint work with Yuan [WY1], we proved an a priori interior Hessian estimate for solutions of (2.2), namely, *the Hessian D^2u of a solution to (2.2) is bounded in terms of the gradient Du.* By known gradient estimates (cf. Trudinger [T]) for σ_k equations, this bounds D^2u in terms of u. One immediate consequence of this estimate is a Liouville type result for global solutions with quadratic growth to (2.2), namely any such a solution must be quadratic (cf. [Y1][Y2]). Another consequence is the regularity (analyticity) of the C^0 viscosity solutions to (2.2), or (2.1) with $n = 3$ and $\Theta = \pm\pi/2$.

In the 1950's, Heinz derived a Hessian bound for the two dimensional Monge-Ampère equation, $\sigma_2(D^2u) = \lambda_1\lambda_2 = \det(D^2u) = 1$, which is equivalent to (2.1) with $n = 2$ and $\Theta = \pm\pi/2$. In the 1970's, Pogorelov constructed counterexamples, namely irregular solutions to three dimensional Monge-Ampère equations $\sigma_3(D^2u) = \lambda_1\lambda_2\lambda_3 = \det(D^2u) = 1$. These counterexamples were generalized for σ_k-equations, with $k > 3$ in higher dimensions by Urbas in 1990 [U2].

The methods involved in this proof directly involve special Lagrangian geometry, in particular, the Michael-Simon mean value and Sobolev inequalities [MS] for minimal submanifolds. We find a function $b(D^2u)$ which is subharmonic (with respect to the induced metric on the Lagrangian graph), so that b at any point is bounded by an integral on the minimal surface, by Michael-Simon's mean value inequality, and further, b satisfies a Jacobi inequality, $|\nabla_g b|^2 \leq \triangle_g b$. This Jacobi inequality, combined with the Sobolev inequality, leads to a bound on the integral of b by the volume of the ball on the minimal surface. The volume element of the minimal Lagrangian graph has a divergence form whose integral lends itself to a bound in terms of the height, Du.

It was our investigations into isoperimetric and mean value type inequalities that led to sharper estimates in the two dimensional case and an estimate for larger phases in the three dimensional case.

2.1.2. *Large phase special Lagrangian equations $|\Theta| \geq (n-2)\pi/2$.* In the joint works with Yuan, [WY2][WY4], we proved the a priori uniform estimate, *any solution to (2.1) with $|\Theta| \geq (n-2)\pi/2$ for $n = 2, 3$ has Hessian D^2u bounded in terms of the gradient Du.* Also, *the gradient Du is bounded by the oscillation of the solution u.* Both estimates are *independent of phase* Θ.

In the 1990's Gregori [G] extended Heinz's estimate to a gradient bound in terms of the heights of the two dimensional minimal surfaces with any codimension. Although it is not clear whether the exponential dependence in our estimate is sharp, still it is sharper than the double exponential dependence on Du by Heinz [H2], [H1] and Gregori, when applied to the special Lagrangian equation of dimension two.

Our methods of proof in these cases follow a similar idea as in the estimate for (2.2), but with some key differences. For $n = 3$ we found that for technical reasons the Michael-Simon Sobolev inequality gave us a quantity which is difficult to control. Instead, by looking directly at isoperimetric inequalities on level sets of subharmonic functions, we were able to derive an important Sobolev type inequality for noncompactly supported functions. This type of argument leads to a sharper bound in the two dimensional case, using only elementary methods.

Also involved was the observation that both gradient and Hessian estimates for special Lagrangian equations of a given phase can be related to the same estimates for equations of a slightly smaller phase, via a Lewy rotation. In particular for $n = 2$, we were able to obtain estimates for small phase special Lagrangian equations from well known estimates for the Laplace equation $\triangle u = 0$. These rotations were necessary in making the estimate uniform, as our primary method gives deteriorating results for small Θ. The gradient estimates (which were proved for general dimension) were obtained by first deriving an estimate in the critical phase $|\Theta| = (n-2)\,\pi/2$, and then applying a Lewy rotation. Most of the proof of a Θ-dependent estimate goes through for general $n \geq 4$, when $|\Theta| > (n-2)\,\pi/2$, but some technical issues remain. In the case $|\Theta| \geq (n-1)\,\pi/2$, the estimates are attainable by our methods, which is shown in the joint work with Chen and Yuan [CWY]. As an application of the results in [CWY], we have regularity of viscosity solutions, and existence for the Dirichlet problem with continuous boundary data.

2.1.3. *Special Lagrangian equations with constraint.* In the joint work with Yuan [WY3], we showed that the pointwise Bernstein-Pogorelov-Korevaar [K] technique can be used to derive Hessian estimates with a specified smallness assumption on the gradient, and a certain convexity constraint. The convexity constraint, $3 + (1-\varepsilon)\lambda_i^2 + 2\lambda_i\lambda_j \geq 0$, gives a Jacobi inequality which holds regardless of dimension or phase. From this we were able to derive a Liouville type result, namely, *any solution to (2.1) satisfying the above convexity constraint and small linear growth on the gradient Du, must be quadratic.* In [CWY] we showed that if the solution u is convex, then *the Hessian D^2u bounded in terms of the gradient Du,* without any smallness assumptions on the gradient. Convexity is an automatic consequence of very large phase $|\Theta| \geq (n-1)\,\pi/2$.

2.2. **Calibrations for a pseudo-Euclidean space.** Also related to the SYZ conjecture, the work of Hitchin [Hi] and McLean [Mc] showed that the moduli space of special Lagrangian submanifolds of a given M can be given a natural metric, and this metric is obtained locally via a special Lagrangian embedding into a pseudo-Euclidean space with signature (n, n). In this pseudo-Euclidean space, special Lagrangian submanifolds arise as solutions to the Monge-Ampère equation,

$$\det(D^2u) = c. \tag{2.4}$$

In 2006 in [Wa], by exhibiting a calibration, I demonstrated a result analogous to that of Harvey and Lawson for $\mathbb{R}^n \times \mathbb{R}^n$ with a pseudo-Euclidean metric of index n, namely, that *gradient graphs of solutions to (2.4) define submanifolds which are volume maximizing among homologous space-like submanifolds.*

3. FUTURE DIRECTIONS

3.1. **Estimates for σ_2 equations.** In light of Pogorelov's counterexamples, an immediate question is if our proof of the estimate for the σ_2 equation can be generalized to obtain estimates hold for the equation $\sigma_2 = 1$ when $n \geq 4$, or for other equations such as $\sigma_2 = f(x)$. If the method generalizes, one must find a way to replace the Michael-Simon inequalities. This is possible in some cases, as we found in studying the 2-d special Lagrangian equations. More generally, it is one of my goals to understand mean value and isoperimetric inequalities associated to nonlinear equations such as $\sigma_2 = 1$ for $n \geq 4$, that are not necessarily minimal surface equations.

3.2. **Special Lagrangian equations of phases $|\Theta| < (n-2)\pi/2$.** To this point, we have been unable to crack this case. Leaving $n \geq 4$ aside for now, a challenging goal is to complete the picture in 3-d by deriving Hessian and gradient estimates for (2.1) with $|\Theta| < \frac{\pi}{2}$. While many of the tools used in the other estimates are still available, a Jacobi type inequality is difficult to show. Heuristically, one can expect some regularity, at least for phases near $\frac{\pi}{2}$, as regularity tends to be an "open" principle. If there are "natural" or optimal conditions (looser than say, the convexity constraint in 2.1.3 above, if possible) under which regularity holds, it would be of interest to find these. At the same time, we have been exploring the possibility of counterexamples to regularity for phase $\Theta = 0$.

3.3. **Interesting examples of special Lagrangian surfaces.** To this point, very few classes of explicit examples of global solutions to (2.1) are known. For $n = 2$, the only possible examples of global solutions are quadratic polynomials or harmonic functions [F]. In fact, Yuan [Y1] showed that the only global solutions with phase $|\Theta| > (n-2)\pi/2$ are quadratic. For $n = 3$, the only global examples to my knowledge arise as a combination of harmonic functions and polynomials, in phases $\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. As there is an abundance of solutions (harmonic functions) to the phase $\Theta = 0$ equation for $n = 2$, it seems likely that there would also be an abundance of interesting solutions to phase $\Theta = \frac{\pi}{2}$ for $n = 3$, but we have been unable to find or construct even one. One possibility is to use our estimates for the σ_2 equation to construct a solution, along the lines of the celebrated construction by Bombieri, De Giorgi, and Giusti of the first known nontrivial graphical minimal surface in $\mathbb{R}^9$.

Even nonglobal solutions, such as Scherk's famous minimal surfaces, would be of interest, as these may shed light on special Lagrangian geometry. We have used Pogorelov's method (recall counterexamples discussed above) to construct nonglobal solutions to (2.2) and (2.3) for $n = 3$, but the solutions do not give complete special Lagrangian graphs.

3.4. **Special Lagrangian submanifolds.** Of major interest since the SYZ conjecture are the special Lagrangian submanifolds of the curved Calabi-Yau manifolds. Important questions are those of existence, regularity, and structure of the moduli space for special Lagrangian submanifolds. From recent works such as [Sch][SchW1][SchW2][SchW3] we see that the topological and geometrical aspects of these questions are related in a nontrivial way. For these applications we would like to adapt our techniques to more general special Lagrangian equations, which are unlikely to have the scalar form (2.1.) We would like to further understand Lagrangian mean curvature flow, which is still a subject early in its development, see [N][S]. A better understanding of the theory for special Lagrangian equations on curved manifolds and for branched and other singularities is needed to begin to answer these important questions with regard to special Lagrangian submanifolds.

The second boundary value problem for special Lagrangian equations is also worth study. For Monge-Ampère equations, the second boundary value problem has seen progress (recently [TW], in the 1990's [C][U]) in connection with the optimal transportation problem (see below), and in connection with special Lagrangian submanifolds when $n = 2$ [W]. The related Lagrangian prescribed boundary problem recently has seen relevant results ([HP], [SchW1]).

3.5. **The optimal transportation problem.** The centuries old optimal transportation problem, which has wide-ranging impact on many fields, including biology, medicine, and physics, has seen recent progress, and has developed connections to Lagrangian geometry. In 2007, Kim and McCann [KMc] used a pseudo-Riemannian metric of index n, in a proof of the regularity of solutions to the optimal transportation problem. A differential condition on the cost function, introduced by Ma, Trudinger, and Wang [MTW] is the natural condition to expect regularity, and in fact necessary as counterexamples by Loeper show [L]. In the setting of [KMc], the optimal map is given by a space-like graph which is Lagrangian with respect to a para-Kähler form, and the MTW condition becomes a sectional curvature condition on the ambient manifold. For the quadratic cost function $c(x, y) = x \cdot y$, the optimal map between two regions in Euclidean space of uniform volume is represented by a (pseudo) special Lagrangian graph, whose potential satisfies (2.4). Together with Kim and McCann, we are now exploring further implications involving Lagrangian geometry.

3.6. **Equations associated to other calibrations.** The two most well known calibrations complex calibration and the special Lagrangian calibration, associated to the Cauchy-Riemann equations and equations (2.1), respectively. As we saw in [Wa], Monge-Ampère equations are also associated to a calibration. There are many other important calibrations, some of which have been associated to equations (for example, see [HL], [Ch]). I am interested in studying properties of such equations, such as solvability and regularity.

References

[C] Caffarelli, Luis, *Boundary regularity of maps with convex potentials II.* Ann. of Math. **144** (1996), 453-496 .

[Ch] Chen, Jingyi, *Complex anti-self-dual connections on a product of Calabi-Yau surfaces and triholomorphic curves.* Commun. Math. Phys. **201** (1999), 217-247 .

[CWY] Chen, Jingyi; Warren, Micah; and Yuan, Yu, *Hessian estimates for convex solutions to special Lagrangian equations.* In preparation.

[F] Fu, Lei, *An analogue of Bernstein's theorem.* Houston J. Math. **24** (1998), 415–419.

[G] Gregori, Giovanni, *Compactness and gradient bounds for solutions of the mean curvature system in two independent variables.* J. Geom. Anal. **4** (1994), no. 3, 327–360.

[HL] Harvey, Reese; Lawson, H. Blaine. Jr., *Calibrated geometry,* Acta Math. **148** (1982), 47–157.

[HP] Haskins, Mark; Pacini, Tommaso. *Obstructions to special Lagrangian desingularizations and the Lagrangian prescribed boundary problem* Geometry and Topology **10** (2006) 1453-1521.

[H1] Heinz, Erhard, *On certain nonlinear elliptic differential equations and univalent mappings.* J. Analyse Math. **5** (1956/1957), 197–272.

[H2] Heinz, Erhard, *On elliptic Monge-Ampère equations and Weyl's embedding problem.* J. Analyse Math. **7** (1959), 1–52.

[Hi] Hitchin, Nigel J., *The moduli space of special Lagrangian submanifolds. Dedicated to Ennio De Giorgi.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)**25** (1997), no. 3-4, 503–515.

[KMc] Kim, Young-Heon; McCann, Robert J., *Continuity, curvature, and the general covariance of optimal transportation.* Preprint available at arXiv.org.

[K] Korevaar, Nick. *An easy proof of the interior gradient bound for solutions to the prescribed mean curvature equation.* Nonlinear functional analysis and its applications, Part 2 (Berkeley, Calif., 1983), 81–89.

[L] Loeper, Gregoire, *On the regularity of maps solutions of optimal transportation problems.* Preprint available at arXiv.org.

[MTW] Ma, Xi-Nan; Trudinger, Neil S.; and Wang, Xu-Jia, *Regularity of potential functions of the optimal transportation problem.* Arch. Rational Mech. Anal. **177** (2005), no. 4, 151–183.

[Mc] McLean, Robert C., *Deformations of calibrated submanifolds,* Comm. Anal. Geom. **6** (1998), no. 4, 705–747.

[MS] Michael, James H.; Simon, Leon M., *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbb{R}^n$. Comm. Pure Appl. Math. **26** (1973), 361–379.

[N] Neves, André *Singularities of Lagrangian mean curvature flow: zero-Maslov class case.* Invent. Math. **168** (2007), no. 3, 449–484.

[P] Pogorelov, Aleksei Vasil'evich, *The Minkowski multidimensional problem.* Translated from the Russian by Vladimir Oliker. Introduction by Louis Nirenberg. Scripta Series in Mathematics. V. H. Winston & Sons, Washington, D.C.; Halsted Press [John Wiley & Sons], New York-Toronto-London, 1978.

[Sch] Schoen, Richard. *Special Lagrangian submanifolds.* Global theory of minimal surfaces, 655–666, Clay Math. Proc., 2, Amer. Math. Soc., Providence, RI, 2005.

[SchW1] Schoen, Richard; Wolfson, Jon *The volume functional for Lagrangian submanifolds.* Lectures on partial differential equations, 181–191, New Stud. Adv. Math., 2, Int. Press, Somerville, MA, 2003.

[SchW2] Schoen, Richard; Wolfson, J. *Minimizing area among Lagrangian surfaces: the mapping problem.* J. Differential Geom. **58** (2001), no. 1, 1–86.

[SchW3] Schoen, Richard; Wolfson, Jon *Minimizing volume among Lagrangian submanifolds.* Differential equations: La Pietra 1996 (Florence), 181–199, Proc. Sympos. Pure Math., 65, Amer. Math. Soc., Providence, RI, 1999.

[S] Smoczyk, Knut *Longtime existence of the Lagrangian mean curvature flow.* Calc. Var. Partial Differential Equations **20** (2004), no. 1, 25–46.

[SYZ] Strominger, Andrew; Yau, Shing-Tung; and Zaslow, Eric, *Mirror symmetry is T-duality.* Nuclear Phys. B **479** (1996), no. 1-2, 243–259.

[T] Trudinger, Neil S., *Weak solutions of Hessian equations.* Comm. Partial Differential Equations **22** (1997), no. 7-8, 1251–1261.

[TW] Trudinger, Neil S.; Wang, Xu-Jia. *On the second boundary value problem for Monge-Ampère type equations and optimal transportation.* Preprint available at arXiv.org.

[U] Urbas, John, *On the second boundary value problem for equations of Monge-Ampère type.* J. Reine Angew. Math. **487** (1997), 115-124.

[U2] Urbas, John, *On the Existence of Nonclassical Solutions for Two Classes of Fully Nonlinear Elliptic Equations* Indiana Univ. Math. J. **39** No. 2 (1990) 355382J.

[Wa] Warren, Micah, *Calibrations associated to Monge-Ampère equations.* Preprint available at arXiv.org.

[WY1] Warren, Micah; Yuan, Yu, *Hessian estimates for the sigma-2 equation in dimension three.* Preprint.

[WY2] Warren, Micah; Yuan, Yu, *Explicit gradient estimates for minimal Lagrangian surfaces of dimension two.* Preprint available at arXiv.org.

[WY3] Warren, Micah; Yuan, Yu, *A Liouville type theorem for special Lagrangian Equations with constraints.* Preprint available at arXiv.org.

[WY4] Warren, Micah; Yuan, Yu, *Hessian and gradient estimates for three dimensional special Lagrangian equations with large phase.* Preprint.

[W] Wolfson, Jon, *Minimal Lagrangian diffeomorphisms and the Monge-Ampère equation.* J. Diff. Geom. **46** (1997) , 335-373.

[Y1] Yuan, Yu, *A Bernstein problem for special Lagrangian equations.* Invent. Math. **150** (2002), 117–125.

[Y2] Yuan, Yu, *Global Solutions to special Lagrangian Equations*, Proc. Amer. Math. Soc. **134** (2006), no. 5, 1355–1358.

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