Practice Test for the Midterm Exam

The Midterm will take place October 31st, during lecture time. It will cover the material of Chapter 1, Chapter 2 and Chapter 3, Section 3.1.
You do not need to bring paper to write on, because there will be enough room on the test paper.
Be reminded that anything you write on the midterm test paper must be your own work. If there is evidence that you are claiming credit for work that is not your own work during the test period, I need to give you a zero on the test and turn the evidence over to the Dean's Committee on Academic Conduct.
Study suggestions: To be successful on the test I highly recommend to catch up on the homework. After that repeat solving problems. Find out with which kind of problems you are still struggling. Study those over and over again, so that you develop a routine. If there are questions that you just cannot find an answer for, do not hestitate to visit me during office hours. I am more than happy to explain whatever is unclear.
Exam advise: Start with whatever problem seems easy to you. Having solved something right away calms down and gives a secure feeling. Do not waste too much time on a specific problem if you get stuck. Rather start with another problem and come back later, if there is time left. I appreciate very much, if your work is layed down in a logic order and in readable writing. If you need more space than is available on the paper, give clear instructions about where to find the rest of your work. Place a box around your final answer.
On the following pages you find review problems. You need to turn your answers in at latest by Monday, 10/27, in order to get points towards the Midterm.
The following set is a set of vectors which might help you with TRUE/FALSE problems, for instance. You can combine vectors or take specific vectors from this set to get a picture for yourself how a certain statement can look like in

a specific example. Otherwise you do not need to work with the set.

$$\{\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\}.$$

For TRUE/FALSE problems, you just need to circle the correct answer, unless you are asked to give a proof/counterexample.

1. Give the definition of *linear independence of vectors* $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$.

   $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$ are linearly independent, if the only solution to

   $$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots + x_m\mathbf{u}_m = \mathbf{0}$$

   is the trivial solution.

2. Proof the following. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, then so are $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$.

   We need to determine, what the solutions for

   $$x_1(\mathbf{u}+\mathbf{v}) + x_2(\mathbf{u}-\mathbf{v}) = \mathbf{0}$$

   are. But this equation can be rewritten as $(x_1+x_2)\mathbf{u} + (x_1-x_2)\mathbf{v} = \mathbf{0}$. By assumption, $\mathbf{u}, \mathbf{v}$ are linearly independent, so the previous equation can only have the trivial solution, i.e. $x_1 + x_2 = 0$ and $x_1 - x_2 = 0$. From the second equation, we get $x_1 = x_2$ and we substitute this in the first equation to get $2x_1 = 0$. Hence $x_1 = x_2 = 0$, so the linear independence follows.

3. Proof the following. If there are six *linearly dependent* vectors in $\mathbb{R}^6$, then these vectors do not span $\mathbb{R}^6$. (Remark: This is not easy!)

   If there is a set that is linearly independent, then by Theorem 2.4.8 one of the vectors, $\mathbf{u}_1$, say, is in the span of the other five, $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5, \mathbf{u}_6\}$. By Theorem 2.3.4, $\text{span}\{\mathbf{u}_1, \ldots \mathbf{u}_6\} = \text{span}\{\mathbf{u}_2, \ldots, \mathbf{u}_6\}$. But there are only five vectors in $\{\mathbf{u}_2, \ldots, \mathbf{u}_6\}$, so by Theorem 2.3.5, they cannot span $\mathbb{R}^6$.

4. Determine, if the following set of vectors are linearly independent and justify your answers:

(a)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

These are four vectors in $\mathbb{R}^3$, so they cannot be linearly independent by Theorem 2.4.7.

(b)

$$\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix} \right\}.$$

We apply Theorem 2.4.3 and perform Gauss elimination with the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & -5 & 0 \\ -3 & 6 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & -5 & 0 \\ 0 & 12 & -15 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -5 & 0 \\ 0 & 4 & -5 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 2 & -5 & 0 \\ 0 & 4 & -5 & 0 \\ 0 & 0 & -21 & 0 \end{array}\right]$$

There are no free variables, so there is only one solution to the homogeneous system. Because the trivial solution is always a solution to a homogeneous system, it is therefore the only one. Hence, the vectors are linearly independent.

(c) TRUE or FALSE: If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly dependent, so is $\{\mathbf{u}_1, \mathbf{u}_2.\}$

This is false in general. Consider the set $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, from which you can take out the zero vector and end up with two linear independent vectors.

(d) TRUE or FALSE: If $\mathbf{u}_4$ is *not* a linear combination of $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is linearly independent.

This is false: Any one of the other vectors could be the zero vector, for example.

(e) TRUE or FALSE: $\{\mathbf{u}\}$ is *always* linearly independent.

This is false, because $\mathbf{u}$ could be the zero vector which is always linearly dependent, see also Theorem 2.4.6.

(f) Give an example of a 4-set of distinct vectors $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \subset \mathbb{R}^3$ that does *not* span $\mathbb{R}^3$.

Consider

$$\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$$

5. Determine, if $\text{span}\{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}\} = \mathbb{R}^3$ and justify your answer.

We apply the Big Theorem 2.4.17 (a)$\rightarrow$(c). So performing Gauss elimination to

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 5 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -7 & 7 \\ 0 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This last matrix shows us that not for every $\mathbf{b} \in \mathbb{R}^3$, there might be a solution. We might encounter an inconsistent system, because there is a zero row. Therefore, we conclude that the columns do not span $\mathbb{R}^3$.

6. (a) Write down an example of a homogeneous linear system with three rows and three variables.

$$\begin{array}{ccccccc} x_1 & & & + & x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 0 \\ 2x_1 & + & 3x_2 & + & 5x_3 & = & 0 \end{array}$$

(b) Write down the corresponding augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 3 & 5 & 0 \end{array}\right]$$

(c) If the columns of the augmented matrix of a homogeneous system are linearly independent, then the system has exactly one solution. If the columns are linearly dependent, then the system has infinitely solutions.

(d) TRUE or FALSE: A homogeneous linear system can not have no solution.

This is true, because the trivial solution is always a solution to a homogeneous system.

(e) TRUE or FALSE: If the columns of a homogeneous system are linearly dependent, then there are infinitely many solutions to the system.

True.

7. Solve the following linear system.

$$\begin{array}{rcrcrcr} -4x_1 & + & 5x_2 & - & 10x_3 & = & 4 \\ x_1 & - & 2x_2 & + & 3x_3 & = & -1 \\ 7x_1 & - & 17x_2 & + & 34x_3 & = & -16 \end{array}$$

We perform Gauss elimination with the associated augmented matrix.

$$\left[\begin{array}{rrr|r} -4 & 5 & -10 & 4 \\ 1 & -2 & 3 & -1 \\ 7 & -17 & 34 & -16 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ -4 & 5 & -10 & 4 \\ 7 & -17 & 34 & -16 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & -3 & 2 & 0 \\ 0 & -3 & 13 & -9 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -2 & 3 & -1 \\ 0 & -3 & 2 & 0 \\ 0 & 0 & 11 & -9 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 0 & 16/11 \\ 0 & -3 & 0 & 18/11 \\ 0 & 0 & 1 & -9/11 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 0 & 4/11 \\ 0 & 1 & 0 & -6/11 \\ 0 & 0 & 1 & -9/11 \end{array}\right]$$

So there is a unique solution given by $(4/11, -6/11, -9/11)$.

8. Consider the function $T : \mathbb{R}^1 \to \mathbb{R}^1, \mathbf{x} \mapsto m\mathbf{x} + b$. Justify your answers for the following questions.

(a) Find the values for m and b, such that T is a homomorphism.

In order for T to be a homomorphism, we need to make sure that the definining properties as of Definition 3.1.1 are satisfied. The second property is $T(c\mathbf{x}) = cT(\mathbf{x})$. We therefore know that *any* linear homomorphism satisfies $T(\mathbf{0}) = T(0 \cdot \mathbf{0}) = 0T(\mathbf{0}) = 0$! For T as defined in this problem we get from this $0 = T(0) \iff 0 = 0 + b \iff b = 0$. Are there restrictions to m? Let us just check: $T(rx_1 + sx_2) = m(rx_1 + sx_2) = mrx_1 + msx_2 = rmx_1 + smx_2 = rT(x_1) + sT(x_2)$ for all $r, s \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^1$. There is therefore no restriction on m in order to have T to be a homomorphism.

(b) Find the values for m and b, such that T is an injective homomorphism.

We know already from (a) that T is of the form $T : \mathbb{R}^1 \to \mathbb{R}^1, \mathbf{x} \mapsto m\mathbf{x}$. If T is injective, then by Theorem 3.1.7, $T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution. So let us check that equation out for this T.

$$T(\mathbf{x}) = \mathbf{0} \iff m\mathbf{x} = \mathbf{0}.$$

There is only one case, when we have more than just the trivial solution $\mathbf{x} = 0$, namely when $m = 0$. So whenever $m \neq 0$, T is an injective homomorphism.