Math 308 C **Midterm, May 8th 2015** **Spring 2015**

Your Name

Your Signature

Student ID #

	1.	2.	3.	4.	5.	Form	Bonus	$\sum$
Points								
Possible	20	21	6	6	16	3	(7)	72

- No books are allowed. You may use a calculator.
- Place a box around $\boxed{\text{your final answer}}$ to each question.
- If you need more room, use the back of each page and indicate to the grader how to find the logic order of your answer.
- Raise your hand if you have questions or need more paper.
- For TRUE/FALSE problems, you just need to cross the right box. For each correct answer, you will get 1 point, for each incorrect answer, -1 point is added. For no answer you will get zero points. In each subsection of the TRUE/FALSE part, you can never get less than zero points.
- In order to receive points for an accurate form, solutions to systems must be written as a set, vectors need to be underscored to distinguish them from scalars and between equivalent matrices theremust not be an equality sign but an arrow or $\sim$.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

1.)(20 points) For each correct answer in the TRUE/FALSE part, you will get 1 point, for each incorrect answer, there will be one point subtracted, i.e. you get -1 point. For no answer, you get 0 points. You can not get less than 0 points out of one subproblem

(a)	Cross the right box for the statements about linear systems.		
	A homogeneous system with 3 variables and 3 equations always has exactly one solution.	☐ TRUE	⊠ FALSE
	A consistent linear system with m variables and n equations where $m > n$ has infinitely many solutions.	⊠ TRUE	☐ FALSE
	Let $A\mathbf{x} = \mathbf{b}$ represent a linear system with A an (n,m)-matrix. Then any solution vector of that system is in $\mathbb{R}^n$.	☐ TRUE	⊠ FALSE
	A homogeneous system is always consistent.	☐ TRUE	☐ FALSE
	The trivial solution is always a solution to a linear system.	☐ TRUE	⊠ FALSE
	If $A\mathbf{x} = \mathbf{0}$ has only one solution $\mathbf{s}$, then $\mathbf{s} \neq \mathbf{0}$ is possible.	☐ TRUE	⊠ FALSE
(b)	Cross the right box for the statements about linear independence and span.		
	$\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \mathbb{R}^3$ is linearly independent for any $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^3$.	☐ TRUE	⊠ FALSE
	$\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2\}$	⊠ TRUE	☐ FALSE
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \mathbb{R}^4$ might span $\mathbb{R}^4$.	☐ TRUE	⊠ FALSE
	For $\mathbf{u}_1, \dots, \mathbf{u}_4 \in \mathbb{R}^n$, $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \subseteq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.	⊠ TRUE	☐ FALSE
(c)	Cross the right box for the statements about linear systems and linear independence.		
	If A is an (n,m)-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$ and the columns of A are linearly independent, then the linear system $A\mathbf{x} = \mathbf{b}$ cannot have free variables.	⊠ TRUE	☐ FALSE
	Two vectors of $\mathbb{R}^n$ are always linearly independent.	☐ TRUE	⊠ FALSE
	If A is an (n,n)-matrix with linearly independent columns, then these columns span $\mathbb{R}^n$.	⊠ TRUE	☐ FALSE
(d)	Cross the right box for the statements about linear homomorphisms		
	If A is an (n,m)-matrix, then the linear homomorphism that maps a vector $\mathbf{u}$ to $A\mathbf{u}$, has $\mathbb{R}^n$ as domain and $\mathbb{R}^m$ as codomain.	☐ TRUE	⊠ FALSE
	If A is an (n,m)-matrix with linearly independent columns, then $T : \mathbf{u} \mapsto A\mathbf{u}$ is an injective homomorphism.	⊠ TRUE	☐ FALSE
	The kernel of a homomorphism is a subset of the codomain of T.	☐ TRUE	⊠ FALSE
	A linear homomorphism $T : \mathbb{R}^{2002} \to \mathbb{R}^{1999}$ cannot be injective, but it can be surjective.	⊠ TRUE	☐ FALSE
	Let A, B be (n,m)-matrices. Then the function $T : \mathbb{R}^m \to \mathbb{R}^n, \mathbf{u} \mapsto (A+B) \cdot \mathbf{u}$ is a linear homomorphism.	⊠ TRUE	☐ FALSE
	For a surjective homomorphism, the codomain is equal to the range of T.	⊠ TRUE	☐ FALSE
	The function that sends every vector of $\mathbb{R}^n$ to the vector $[1, 1, \dots, 1]^t$ of $\mathbb{R}^m$ is a linear homomorphism.	☐ TRUE	⊠ FALSE

2.(3+6+2+1+4+1+(2+2)) Consider the following linear system:

$$\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & + & 2x_3 & & & = & 6 \\ 2x_1 & + & 4x_2 & - & x_3 & & & = & -3 \\ -1x_1 & - & x_2 & - & x_3 & - & 2x_4 & = & -2 \\ 3x_1 & - & 3x_2 & + & 2x_3 & - & 2x_4 & = & 7 \end{array}$$

(a) Find the coefficient matrix A and the augmented matrix B for this system.

$$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 2 & 4 & -1 & 0 \\ -1 & -1 & -1 & -2 \\ 3 & -3 & 2 & -2 \end{bmatrix}$$

$$B = \left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & 6 \\ 2 & 4 & -1 & 0 & -3 \\ -1 & -1 & -1 & -2 & -2 \\ 3 & -3 & 2 & -2 & 7 \end{array}\right]$$

(b) Perform the Gauss-Jordan algorithm on the augmented matrix B.

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & 6 \\ 2 & 4 & -1 & 0 & -3 \\ -1 & -1 & -1 & -2 & -2 \\ 3 & -3 & 2 & -2 & 7 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & 6 \\ 0 & 10 & -5 & 0 & -15 \\ 0 & -4 & 1 & -2 & 4 \\ 0 & 6 & -4 & -2 & -11 \end{array}\right]$$

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & 6 \\ 0 & 2 & -1 & 0 & -3 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & -1 & -2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -3 & 2 & 0 & 6 \\ 0 & 1 & -1/2 & 0 & -3/2 \\ 0 & 0 & -1 & -2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{rrrr|r} 1 & -3 & 0 & -4 & 2 \\ 0 & 1 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 1/2 \\ 0 & 1 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) The matrix obtained in (b), i.e. after Gauss Jordan, is called matrix in

reduced echelon form.

(d) Identify the free variables of that system if any exist. The variable x_4 is a free variable.

(e) Find the solutions of the system in vector form. Write a proper solution set with declaring all free parameters as your final answer. The solution set is as follows:

$$S = \{ \begin{bmatrix} 1/2 \\ -1/2 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \end{bmatrix}, t \in \mathbb{R}\}$$

(f) Based on your answer in (d) or (e), how many solutions does the system have?
As there is a free variable, the system has infinitely many solutions.
(g) In class we had the theorem which stated that any general solution $\mathbf{u}_g$ of an inhomogeneous system can be written as $\mathbf{u}_g = \mathbf{u}_p + \mathbf{u}_h$, where $\mathbf{u}_p$ is a particular (fixed!) solution and $\mathbf{u}_h$ is a solution to the homogeneous system $A\mathbf{x} = \mathbf{0}$ (with A as in (a)).

(i) How many solutions does $A\mathbf{x} = \mathbf{0}$ have? Use By (f) and the mentioned theorem and explain how you came to your conclusion.

As $\mathbf{x}_p$ is a fixed (sibgle soultion), the number of solutions to the homogeneous system must be equal to the number of solutions to the system we just considered. As there are infinitely many, the associated homogeneous system also has infinitely many solutions.

(ii) Are the columns of A linearly independent? Justify your answer.

As $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, it has not only the trivial solution. But this means that the columns are linearly dependent. (Big Theorem.)

3.(4+2)

(a) Determine h, so that the following vectors are linearly independent:

$$\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 2h \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & -3 & 2h & 0 \end{array}\right] \quad \rightarrow \quad \left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & -1 & 4 & - \\ 0 & -2 & 2h+6 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 2h-2 & 0 \end{array}\right]$$

In order to achieve linear independence, we need to make sure that the system does not have any free variables (because we only like to see the trivial soulution.) We don't have any free variables, if the last row is not a zero row. Therefore

$$h \neq 1.$$

(b) Let $h = 1$. Apply the Big Theorem and conclude whether the vectors span $\mathbb{R}^3$ in this case.

As we have linear dependence for $h = 1$ by (a), we must negate the Big Theorem, which means, that the columns do not span $\mathbb{R}^3$ in this case.

4.(3+3)

(a) Write down the precise definition of 'Linear Independence' of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$. The definition must be written in whole sentences and must include the expressions 'linear independent, equation, solution, only'.

The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$ is linearly independent, if and only if the vector equation

$$\{c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \ldots + c_m\mathbf{u}_m = \mathbf{0}$$

only has the trivial solution.

(b) Let $\mathbf{u}$ be a vector in $\mathbb{R}^k$ and let A be an (m, n)-matrix. How must k be chosen so that the matrix vector product $A\mathbf{u}$ is defined? In which Euclidean space does $A\mathbf{u}$ lie? (Don't forget to give both answers!)

For an (m, n)-matrix A, the vector product is only defined if the number of columns of A (which is n) is equal to the number of rows of $\mathbf{u}$. So $k = n$. The result is a vector in $\mathbb{R}^m$.

5.(3+4+2+3+4) Consider the following linear homomorphism:

$$T : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \rightarrow \begin{bmatrix} u_1 - u_2 - u_3 \\ u_1 \\ 2u_1 + 2u_3 \end{bmatrix}.$$

(a) Find the corresponding matrix A, such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

(b) Find the kernel of T. (The answer is a set of vector(s)!)
We need to find all elements of the domain which are mapped to the zero vector. We therefore need to solve (remember "our" interpretations.)

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

This shows that $x_3 = 0, x_2 = 0, x_1 = 0$, so the only solution is the zero vector. Therefore, $\ker(T) = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \}$.

(c) Based on your answer in (b), is T injective? Justify your answer.
Because the only element in the kernel is the zero vector, we have injectivity.

(d) Find the image of the vector $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ under T.

The image under $T$is found by multiplying the matrix and the vector.

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 8 \end{bmatrix}$$

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(e) Find all vectors $\mathbf{u} \in \mathbb{R}^3$, such that $T(\mathbf{u}) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

We need to find the preimage of the vector, i.e. we have to solve $A\mathbf{x} = \mathbf{u}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Backward substitution gives $x_3 = 0, x_2 = 1, x_1 = 1$, so the only element that is mapped to the given vector is

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$