Math 308 **Practice Test for Final Exam** **Winter 2015**

- No books are allowed during the exam. But you are allowed one sheet (10 x 8) of handwritten notes (back and front). You may use a calculator.
- For TRUE/FALSE problems, you just need to cross the right box. For each correct answer, you will get 1 point, for each incorrect answer, -1 point is added. For no answer you will get zero points. In each subsection of the TRUE/FALSE part, you can never get less than zero points.
- Only the sections that we covered in class will be relevant. !!So, if there is a problem that we is related to a topic we might not get in class, you will not need to know that for the exam!!
- You can earn bonus points for turning this practice test in by March 13th AND by doing the online evalution for this course. All test that are turned in by March 10th will be checked.
- In the exam there will be points for formal correctness as we have already seen in the Midterm.
- You will be given the chance to make up some points after the exam. Be sure to check your emails frequently after the exam, so that you will be able to use that chance.

I certify that I have evaluated this course online. Your Signature

1.)

(a)	Cross the right box for the statements about linear systems.		
	There are homogeneous systems with no solution.	☐ TRUE	☒ FALSE
	Every linear system with more variables than equations has at least one solution.	☐ TRUE	☒ FALSE
	A linear system of the form $A\mathbf{x} = \mathbf{b}$ with A an (n, m)-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$ has a solution, if and only if $\mathbf{b}$ is in the column space of A.	☒ TRUE	☐ FALSE
	Let $\mathbf{u}_i$ be vectors in $\mathbb{R}^n$. Then the linear system $[\mathbf{u}_1\ \mathbf{u}_2\ \dots\ \mathbf{u}_n \vert \mathbf{u}_2]$ has $[0, 1, 0, \dots, 0]^t$ as a solution.	☒ TRUE	☐ FALSE
	Any homogeneous system with 5 variables and 3 equations has infinitely many solutions.	☒ TRUE	☐ FALSE
	Let $\mathbf{s} \in \mathbb{R}^m$ and A an (n, m)-matrix such that $A\mathbf{s} = \mathbf{b}$. Then $\mathbf{s}$ is a solution to $A\mathbf{x} = \mathbf{b}$.	☒ TRUE	☐ FALSE
	If $A\mathbf{x} = \mathbf{b}$ has a solution and $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then $A\mathbf{x} = \mathbf{b}$ has also infinitely many solutions.	☒ TRUE	☐ FALSE
(b)	Cross the right box for the statements about linear independence, span and bases.		
	Including the zero vector in $\mathbb{R}^n$ always gives a linearly dependent set of vectors in $\mathbb{R}^n$.	☒ TRUE	☐ FALSE
	Let $\{\mathbf{u}_1, \dots \mathbf{u}_n\} \subseteq \mathbb{R}^m$. Then $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_n\} \subsetneq \{\mathbf{u}_1, \mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_2, \mathbf{u}_3, \dots \mathbf{u}_n\}$	☐ TRUE	☒ FALSE
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} \subseteq \mathbb{R}^4$ is always linearly dependent.	☒ TRUE	☐ FALSE
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\} \subseteq \mathbb{R}^4$ always spans $\mathbb{R}^4$.	☐ TRUE	☒ FALSE
	If S is a subspace of $\mathbb{R}^4$ of dimension 3, then any basis of S must contain exactly 3 vectors of $\mathbb{R}^4$.	☒ TRUE	☐ FALSE
	Every subspace has a unique basis.	☐ TRUE	☒ FALSE
	If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbb{R}^3$, then so does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, for a vector $\mathbf{u}_4 \in \mathbb{R}^3$.	☒ TRUE	☐ FALSE
(c)	Cross the right box for the statements about linear independence, span, bases.		
	Let $\mathbf{u} \in \mathbb{R}^m$ and $a \in \mathbb{R}$. Then $\{\mathbf{u}, a\mathbf{u}\}$ is linearly dependent.	☒ TRUE	☐ FALSE
	If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ spans $\mathbb{R}^4$, then the set must be a basis of $\mathbb{R}^4$.	☒ TRUE	☐ FALSE
	There cannot be a basis that contains the zero vector.	☒ TRUE	☐ FALSE
	The only subspace of $\mathbb{R}^n$ with dimension n is $\mathbb{R}^n$.	☒ TRUE	☐ FALSE
	Let S_1, S_2 be two subspaces of $\mathbb{R}^n$ with $S_1 \subseteq S_2$. Then $\dim(S_1) \leq \dim(S_2)$.	☒ TRUE	☐ FALSE
	Let S be a subspace of $\mathbb{R}^7$ with $\dim(S) = 3$. Then there is a linearly independent set $U \subseteq S$ with 4 elements.	☐ TRUE	☒ FALSE
	The number of elements in a basis of a subspace $S \neq \{\mathbf{0}\}$ is always equal to the dimension of S.	☒ TRUE	☐ FALSE
	If $S = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, then $\dim(S) = 3$.	☐ TRUE	☒ FALSE
	Assume that $S \subseteq \mathbb{R}^2$ is a subspace. Then the possible dimensions of S are 0,1 and 2.	☒ TRUE	☐ FALSE
	$S_1 = \text{span}\{[1, 2]^t\}$ and $S_2 = \text{span}\{[1, 3]^t\}$ both are one-dimensional subspaces of $\mathbb{R}^2$, but they are not equal.	☒ TRUE	☐ FALSE

(d)	Cross the right box for the statements about matrices and homomorphisms.		
	If $T : \mathbb{R}^3 \to \mathbb{R}^7$ is a linear homomorphism, then the corresponding matrix A_T such that $T(\mathbf{x}) = A_T\mathbf{x}$ is a $(3,7)$-matrix.	☐ TRUE	☒ FALSE
	If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear homomorphism with corresponding matrix A_T, then $\ker(T) = \text{null}(A_T)$.	☒ TRUE	☐ FALSE
	The matrix product $A \cdot B$ for a $(3,4)$-matrix A and $(3,4)$-matrix B is defined.	☐ TRUE	☒ FALSE
	$T : \mathbb{R}^3 \to \mathbb{R}^3$ is injective and surjective, if and only if the corresponding matrix A_T is invertible.	☒ TRUE	☐ FALSE
	If A is a $(5,13)$-matrix, then the corresponding linear homomorphism $T(\mathbf{x}) = A\mathbf{x}$ has $\mathbb{R}^{13}$ as domain and $\mathbb{R}^5$ as codomain.	☒ TRUE	☐ FALSE
	Let A, B be arbitrary (n,n)-matrices. Then the following equation is always valid: $(A+B)^2 = A^2 + 2A \cdot B + B^2$.	☐ TRUE	☒ FALSE
	If $T : \mathbb{R}^7 \to \mathbb{R}^{17}$ is injective, then T is invertible.	☐ TRUE	☒ FALSE
	Let $T : \mathbb{R}^5 \to \mathbb{R}^8$ be a linear homomorphism. Then $\ker(T)$ is a subspace of $\mathbb{R}^8$.	☐ TRUE	☒ FALSE
(e)	Cross the right box for the statements about the determinant of an (n,n)-matrix A.		
	If $\det(A) = 0$, then A has no inverse.	☒ TRUE	☐ FALSE
	If A and B are invertible, then so is $A \cdot B$.	☒ TRUE	☐ FALSE
	If all the cofactors of A are nonzero, then $\det(A) \neq 0$.	☐ TRUE	☒ FALSE
	If the columns of A are linearly independent, then $\det(A) \neq 0$.	☒ TRUE	☐ FALSE
	The determinant is only defined for square matrices.	☒ TRUE	☐ FALSE
	$\text{rank}(A) = n$ if and only if $\det(A) \neq 0$.	☒ TRUE	☐ FALSE
	A^2 is the zero matrix if and only if A is the zero matrix.	☐ TRUE	☒ FALSE
	If A^2 is the zero matrix, then $\det(A) = 0$	☒ TRUE	☐ FALSE
(f)	Cross the right box for the statements about column- and rowspace of a matrix A.		
	If A is a square matrix, then $\text{col}(A) = \text{row}(A)$.	☐ TRUE	☒ FALSE
	If A is a matrix, then $\dim(\text{col}(A)) = \dim(\text{row}(A))$.	☒ TRUE	☐ FALSE
	If $A\mathbf{x} = \mathbf{b}$ is a consistent system, then $\mathbf{b}$ is an element of $\text{col}(A)$.	☒ TRUE	☐ FALSE
	Let A be a $(4,13)$-matrix. Then $\text{nullity}(A)$ could be equal to 5.	☐ TRUE	☒ FALSE
(g)	Cross the right box for the statements about eigenvalues and eigenspaces of a matrix A.		
	The zero vector $\mathbf{0}$ is an eigenvector for the eigenvalue $\lambda = 0$, because $A\mathbf{0} = 0 \cdot \mathbf{0}$.	☐ TRUE	☒ FALSE
	If 0 is an eigenvalue for A, then $\det(A) = 0$.	☒ TRUE	☐ FALSE
	If A is a $(2,2)$-matrix with the only eigenvalue 0, then A is the zero-matrix.	☐ TRUE	☒ FALSE
	It is possible for A to have no eigenvalue over $\mathbb{R}$.	☒ TRUE	☐ FALSE
(h)	Cross the right box for the statements about orthogonality of vectors.		
	If $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$ and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$, then $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$.	☐ TRUE	☒ FALSE
	If $\mathbf{u}, \mathbf{v}$ both have only nonnegative entries, then $\mathbf{u} \cdot \mathbf{v} \geq 0$.	☒ TRUE	☐ FALSE

2. Consider the following linear homomorphism:

$$T : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \to \begin{bmatrix} u_1 + 2u_2 - u_3 \\ -4u_1 - 7u_2 + 7u_3 \\ -u_1 - u_2 + 5u_3 \end{bmatrix}.$$

(a) Find the corresponding matrix A, such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

$$\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}$$

(b) Find the kernel of T.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -4 & -7 & 7 & 0 \\ -1 & -1 & 5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore,

$$\ker(T) = \text{span}\{[0,0,0]^t\}.$$

(c) Calculate the determinant of A. Is T invertible?

$$\det(A) = -35 - 14 - 4 + 7 + 7 + 40 = 1,$$

so A is invertible.

(d) Find the inverse T^{-1} of T using A.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 7 & 0 & 1 & 0 \\ -1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & 01 & 1 \end{array}\right] \sim,$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -28 & -9 & 7 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right]$$

so the inverse is

$$\begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -3 & -1 & 1 \end{bmatrix}.$$

(e) Do the columns of A form a basis of $\mathbb{R}^3$? Give an explanation for your answer or cite a suitable theorem.

They form a basis because of the Big Theorem and (b) or (c) of this problem.

(f) Let $\mathbf{u} = [0,\ 10,\ 12]^t$ be an element of the range of T. Find the uniquely determined element $\mathbf{v}$ in the domain of T, such that $T(\mathbf{v}) = \mathbf{u}$, using A^{-1}.

$$\begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -4 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ 12 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$$

3. Consider the following matrix:

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 & -1 \\ 2 & -4 & 7 & -3 & 3 \\ 3 & -6 & 8 & 3 & -8 \end{bmatrix}.$$

(a) Find the null space of A.

$$\left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & -1 & 0 \\ 2 & -4 & 7 & -3 & 3 & 0 \\ 3 & -6 & 8 & 3 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 3 & 0 & -1 & 0 \\ 0 & 0 & 1 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Setting $x_2 = s_1, x_4 = s_2, x_5 = s_3$, we get by backward substitution

$$\text{null}(A) = \text{span}\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 16 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \}.$$

(b) Find a basis of null(A), that contains the vector $v = [5,\ 0,\ -1,\ 3,\ 2]^t$.

$$\begin{bmatrix} 5 & 2 & -9 & 16 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 3 & -5 \\ 3 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 5 & 2 & -9 & 16 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which shows, that the first, second and third columns are pivot columns, so that we take the respective vectors and conclude that

$$\{ \begin{bmatrix} 5 \\ 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} \}$$

is a basis as asked for.

(c) What is the nullity of A? What is the rank of A?

The basis in (b) has three elements, so nullity(A) = 3. By the rank-nullity theorem, we therefore have rank(A) = 2.

4. Consider the following matrix:

$$\begin{bmatrix} 0 & -3 & -1 \\ -1 & 2 & 1 \\ 3 & -9 & -4 \end{bmatrix}$$

(a) Determine the characteristic polynomial χ_A of A. Show all your work! (Key: The characteristic polynomial is $\chi_A = -\lambda^3 - 2\lambda^2 - \lambda$.)

$$|A - \lambda \mathbf{I}| = \left| \begin{bmatrix} -\lambda & -3 & -1 \\ -1 & 2-\lambda & 1 \\ 3 & -9 & -4-\lambda \end{bmatrix} \right| = \lambda(2-\lambda)(4+\lambda) - 9 - 9 + 3(2-\lambda) + 9(-\lambda) + 3(4+\lambda) = \lambda^3 - 2\lambda^2 - \lambda.$$

(b) Determine the eigenvalues for A.

$$\chi_A = -\lambda(\lambda^2 + 2\lambda + 1) = -\lambda(\lambda+1)^2 = 0 \iff \lambda = 0 \text{ or } \lambda = -1.$$

(c) Compute the eigenspace for the eigenvalue that has multiplicity 2 in χ_A. What is the dimension of this eigenspace?

$$A - (-1)\mathbf{I} = \left[\begin{array}{ccc|c} 0-(-1) & -3 & -1 & 0 \\ -1 & 2-(-1) & 1 & 0 \\ 3 & -9 & -4-(-1) & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ -1 & 3 & 1 & 0 \\ 3 & -9 & -3 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Setting $x_2 = s_1, x_3 = s_2$, we get $x_1 = 3s_1 + s_2$. We therefore have as the eigenspace

$$E_{-1} = \text{span}\{ \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}.$$

The dimension of this eigenspace is 2.

(d) What is the only possible dimension of the eigenspace with eigenvalue $\lambda = 0$? Answer this question with the help of χ_A and justify your answer. The multiplicity of 0 as a root of χ_A is 1. By Theorem 6.1.11, the dimension of E_0 must be less than or equal to 1, hence it is equal to 1.

5. Let $S = \text{span}\{\mathbf{s}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \mathbf{s}_2 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}\}$.

(a) Find an orthogonal basis using Gram-Schmidt.

Following Gram-Schmidt, we get

$$\mathbf{u}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix} - \frac{-2+12-1}{9} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$$

and thus,

$$\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \}$$

as an orthogonal basis of S.

(b) Find a basis for $S^\perp$.

We need to determine the following nullspace.

$$\left[\begin{array}{ccc|c} 1 & 6 & -1 & 0 \\ -2 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 6 & -1 & 0 \\ 0 & 14 & -1 & \end{array}\right]$$

Setting $x_3 = s$, we get $x_2 = \frac{1}{14}s$ and $x_1 = \frac{8}{14}s$. Hence,

$$S^\perp = \text{span}\{ \begin{bmatrix} 8 \\ 1 \\ 14 \end{bmatrix} \}.$$

(c) Compute the norm $||\mathbf{s}_2||$ of $\mathbf{s}_2$.

$$||\mathbf{s}_2|| = \sqrt{1 + 36 + 1} = \sqrt{38}.$$

(d) What is the exact norm of

$$\frac{1}{||\mathbf{s}_2||}\mathbf{s}_2?$$

$$||\frac{1}{||\mathbf{s}_2||}\mathbf{s}_2|| = \frac{1}{||\mathbf{s}_2||}||\mathbf{s}_2|| = 1.$$

6. Suppose that A is a $(7, 17)$-matrix.
(a) What is the maximum possible value for $\text{rank}(A)$?
The maximum possible value for the dimension of the col-/or rowspace is 7.
(b) What is the minimum possible value for $\text{nullity}(A)$?
By the rank-nullity theorem, we have a minimum possible nullity of 10.
(c) Suppose that $\dim(\text{col}(A)) = 5$. What is $\text{nullity}(A)$?
The rank-nullity theorem we get $\text{nullity}(A) = 12$
7. Let A be a (2,2)-matrix, which has no eigenvalue over $\mathbb{R}$. What is a possible characteristic polynomial for A in this case?
A possible characteristic polynomial without a root over $\mathbb{R}$ is $\chi_A = \lambda^2 + 1$, for example.
8. Show that if $\mathbf{u}_1, \mathbf{u}_2$ are both orthogonal to $\mathbf{v}$, then $\mathbf{u}_1 + \mathbf{u}_2$ is also orthogonal to $\mathbf{v}$.

$$(\mathbf{u}_1 + \mathbf{u}_2)\mathbf{v} = \mathbf{u}_1\mathbf{u} + \mathbf{u}_2\mathbf{v} = 0 + 0 = 0,$$

hence the desired orthogonality.
9. Give examples for infinitely many different subspaces of dimension one in $\mathbb{R}^2$.
For example, $\text{span}\{\begin{bmatrix} 1 \\ n \end{bmatrix}\}$ for $n \in \mathbb{N}$ are examples of infinitely many one dimensional subspaces of $\mathbb{R}^2$.
10. Calculate

$$\left(\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}\right)^2.$$

$$\left(\begin{bmatrix} 2 & 1 \\ -4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}\right)^2 = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}^2 = \begin{bmatrix} 4 & 0 \\ -8 & 4 \end{bmatrix}.$$

11. (a) Let $\mathcal{B}_1 = \{\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \end{bmatrix}\}$ be a basis for $\mathbb{R}^2$ and let $\mathbf{u} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$ be a vector represented with respect to the standard matrix. What is the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}_1$.

The change-of-base-matrix is

$$U = \begin{bmatrix} -2 & 4 \\ 1 & -3 \end{bmatrix},$$

so

$$U^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}.$$

Hence,

$$\mathbf{u}_{\mathcal{B}_1} = \frac{1}{2}\begin{bmatrix} -3 & -4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

(b) Find the missing vector $\mathbf{v}_2$ in the basis $\mathcal{B}_2 = \{\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \mathbf{v}_2\}$ such that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{B}_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Let the missing vector be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The change-of-base formula gives

$$\begin{bmatrix} 4 & v_1 \\ -2 & v_2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

This gives a system with two linear equations

$$\begin{aligned} 4 + 3v_1 &= -2 \\ -2 + 3v_2 &= 1, \end{aligned}$$

with $v_1 = -2$ and $v_2 = 1$ as solution. Hence $\mathcal{B}_2 = \{\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}\}$ is the desired basis.

12. Find the values for h such that A is not invertible, where

$$A = \begin{bmatrix} 0 & 4 & h \\ h & 1 & 3 \\ 0 & h & 1 \end{bmatrix}.$$

$$det(A) = h^3 - 4h = h(h-2)(h+2).$$

In order for A not to be invertible, the determinant must be zero. Hence $h = 0, 2, -2$ are the values such that A is not invertible.

13. (a) Let

$$A = \begin{bmatrix} 2 & 1 & 4 & -2 & 0 & 1 & 2 & 3 \\ 3 & 1 & 5 & -3 & 1 & 0 & 4 & 5 \\ 6 & 3 & 13 & -6 & -3 & -1 & 0 & 1 \\ 0 & 4 & 9 & -1 & 1 & 0 & -10 & -1 \end{bmatrix}.$$

Show **without** determining the nullspace of A that the vector $[1, -5, 1, 0, 1, 1, -1, 0]^t$ is an element of the nullspace of A.

If $\mathbf{v}$ is in null(A), satisfies by definition $A\mathbf{v} = \mathbf{0}$. We therefore check that

$$\begin{bmatrix} 2 & 1 & 4 & -2 & 0 & 1 & 2 & 3 \\ 3 & 1 & 5 & -3 & 1 & 0 & 4 & 5 \\ 6 & 3 & 13 & -6 & -3 & -1 & 0 & 1 \\ 0 & 4 & 9 & -1 & 1 & 0 & -10 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 1 \\ 0 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) Show that the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are orthogonal, if and only if the vector $\mathbf{v}$ is an element of the nullspace of $\mathbf{u}^t$, when $\mathbf{u}^t$ is considered as an $(n, 1)$-matrix.

$\mathbf{u}, \mathbf{v}$ are orthogonal, if and only if $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}^t\mathbf{v} = [0]$, if and only if $\mathbf{v}$ is in the nullspace of $\mathbf{u}^t$.