Math 308 **Midterm, Feb 11th 2015** **Winter 2015**

Your Name

Your Signature

Student ID #

	1.	2.	3.	4.	5.	Form	Bonus	$\sum$
Points								
Possible	20	16	6	6	12	3	(6)	63

- No books are allowed. You may use a calculator.
- Place a box around $\boxed{\text{your final answer}}$ to each question.
- If you need more room, use the back of each page and indicate to the grader how to find the logic order of your answer.
- Raise your hand if you have questions or need more paper.
- For TRUE/FALSE problems, you just need to cross the right box. For each correct answer, you will get 1 point, for each incorrect answer, -1 point is added. For no answer you will get zero points. In each subsection of the TRUE/FALSE part, you can never get less than zero points.
- In order to receive points for an accurate form, solutions to systems must be written as a set, vectors need to be underscored to distinguish them from scalars and between equivalent matrices there is no equality sign but an arrow.

Do not open the test until everyone has a copy and the start of the test is announced.

GOOD LUCK!

1.)(20 points) For each correct answer in the TRUE/FALSE part, you will get 1 point, for each incorrect answer, there will be one point subtracted, i.e. you get -1 point. For no answer, you get 0 points. You can not get less than 0 points out of one subproblem

(a)	Cross the right box for the statements about linear systems.		
	A homogeneous system with 5 variables and 3 equations always has infinitely many solutions.	☒ TRUE	☐ FALSE
	A linear system with 5 variables and 3 equations always has infinitely many solutions.	☐ TRUE	☒ FALSE
	If a linear system of the form $A\mathbf{x} = \mathbf{b}$, with A an (n, m)-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$, has a solution, then this solution can be written as a vector in $\mathbb{R}^m$.	☒ TRUE	☐ FALSE
	Let $\mathbf{u}_i$ be vectors in $\mathbb{R}^n$. Then the linear system with augmented matrix $[\mathbf{u}_1\ \mathbf{u}_2\ \dots\ \mathbf{u}_n \vert \mathbf{u}_2 + \mathbf{u}_3]$ has $[0, 1, 1, \dots, 0]^t$ as a solution.	☒ TRUE	☐ FALSE
	A homogeneous system is always consistent.	☒ TRUE	☐ FALSE
	The trivial solution is always a solution to a linear system.	☐ TRUE	☒ FALSE
	If $A\mathbf{x} = \mathbf{0}$ has only one solution, this must be the trivial solution.	☒ TRUE	☐ FALSE
(b)	Cross the right box for the statements about linear independence and span.		
	$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\} \subseteq \mathbb{R}^3$ spans $\mathbb{R}^3$.	☐ TRUE	☒ FALSE
	$\text{span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{span}\{\mathbf{u}_1 + \mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2\}$	☒ TRUE	☐ FALSE
	If $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent, then there exists a scalar $c \in \mathbb{R}$ such that $\mathbf{u}_1 = c\mathbf{u}_2$ or $\mathbf{u}_2 = c\mathbf{u}_1$.	☒ TRUE	☐ FALSE
	If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ spans $\mathbb{R}^3$, then so does $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$, for any vector $\mathbf{u}_4 \in \mathbb{R}^3$.	☒ TRUE	☐ FALSE
(c)	Cross the right box for the statements about linear systems and linear independence.		
	If A is an (n, m)-matrix and $\mathbf{b}$ a vector in $\mathbb{R}^n$ and the columns of A are linearly independent, then the linear system $A\mathbf{x} = \mathbf{b}$ cannot have free variables.	☒ TRUE	☐ FALSE
	A set of one vector is always linearly independent.	☐ TRUE	☒ FALSE
	If $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are pairwise linearly independent, then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also linearly independent.	☐ TRUE	☒ FALSE
	A homogeneous system with 3 variables and 3 equations has exactly one solution.	☐ TRUE	☒ FALSE
(d)	Cross the right box for the statements about linear homomorphisms		
	If A is an (n, m)-matrix, then the linear homomorphism that maps a vector $\mathbf{u}$ to $A\mathbf{u}$ has $\mathbb{R}^m$ as domain and $\mathbb{R}^n$ as codomain.	☒ TRUE	☐ FALSE
	Any linear homomorphism maps the zero vector of the domain to the zero vector of the codomain.	☒ TRUE	☐ FALSE
	Let T be a homomorphism. Then T is surjective if and only if the zero vector of the domain is the only vector that is mapped to the zero vector of the codomain.	☐ TRUE	☒ FALSE
	A linear homomorphism $T : \mathbb{R}^{12} \to \mathbb{R}^7$ cannot be injective, but it can be surjective.	☒ TRUE	☐ FALSE
	The function that sends every vector of $\mathbb{R}^n$ to the zero vector of $\mathbb{R}^m$ is a linear homomorphism.	☒ TRUE	☐ FALSE

2.(1+3+2+4+1+1+1+3) Consider the following linear system:

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & -2 \\ 4x_1 & + & 5x_2 & - & 3x_3 & = & -2 \\ -x_1 & + & 3x_2 & + & 5x_3 & = & -8 \end{array}$$

(a) Is this an inhomogeneous or a homogeneous system?
This is an inhomoegeneous system
(b) Before actually solving the system, what are the possible numbers of solutions?
There are none, exactly one or infinitely many solutions
(c) What is the corresponding augmented matrix?

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -2 \\ 4 & 5 & -3 & -2 \\ -1 & 3 & 5 & -8 \end{array}\right]$$

(d) Perform the Gauss algorithm to put this matrix into echelon form.

$$\left[\begin{array}{rrr|r} 2 & 3 & -1 & -2 \\ 4 & 5 & -3 & -2 \\ -1 & 3 & 5 & -8 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 2 & 3 & -1 & -2 \\ 0 & -1 & -1 & 2 \\ -0 & 9 & 9 & -18 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 2 & 3 & -1 & -2 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(e) Identify the leading variables.
The leading variables are x_1, x_2.
(f) Are there free variables? If so, identify them.
Yes, there is one free variable, which is x_3.
(g) Based on your answer in (f), what is the number of solutions to that system?
As there is a free variable, there are infinitely many solutions.
(h) Solve the system. Write the solution in vector form (i.e. $\mathbf{x} = \mathbf{v} + s\mathbf{w}$, where you need to specify $\mathbf{v}$ and $\mathbf{w}$). Write a proper solution set with declaring all possible parameters.
By backward substitution we find as solution set

$$S = \{ \left[\begin{array}{r} 2 \\ -2 \\ 0 \end{array}\right] + t \left[\begin{array}{r} 2 \\ -1 \\ 1 \end{array}\right] \mid t \in \mathbb{R}\}$$

3.(4+2)
(a) Determine h, so that the following vectors span $\mathbb{R}^3$:

$$\left[\begin{array}{r} 1 \\ -3 \\ 2 \end{array}\right], \left[\begin{array}{r} 0 \\ 1 \\ h \end{array}\right], \left[\begin{array}{r} -3 \\ 1 \\ 2 \end{array}\right]$$

We need to choose h so that any element of $\mathbb{R}^3$ is in the span of the vectors. Hence, the following system must always be consistent.

$$\left[\begin{array}{rrr|c} 1 & 0 & -3 & * \\ -3 & 1 & 1 & * \\ 2 & h & 2 & * \end{array}\right] \rightarrow \left[\begin{array}{rrr|c} 1 & 0 & -3 & * \\ 0 & 1 & -8 & * \\ 0 & h & 8 & * \end{array}\right] \rightarrow \left[\begin{array}{rrc|c} 1 & 0 & -3 & * \\ 0 & 1 & -8 & * \\ 0 & 0 & 8h+8 & * \end{array}\right]$$

In order to have the system consistent, we see from the equivalent system in echelon form that $-8h+8 \neq 0$, so that $h \neq -1$.
(b) If you choose some h so that the three vectors do span $\mathbb{R}^3$, apply the Big Theorem and conclude the right statement about linear independence of these vectors.
The conditions for the Big Theorem are satisfied. Therefore, we conclude from the fact that the vectors span $\mathbb{R}^3$ that they are also linearly independent.
4.(3+3)
(a) Write down the precise definition of 'Linear Independence' of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$. The definition must be written in whole sentences and must include the expressions 'linearly independent, equation, solution, only'.
$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\} \subseteq \mathbb{R}^n$ is linearly independent, if the only solution to the vector equation

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + \ldots + x_m\mathbf{u}_m = \mathbf{0}$$

is the trivial solution.
(b) Let $\mathbf{u}$ be a vector in $\mathbb{R}^k$ and let A be an (m,n)-matrix. How must k be chosen so that the matrix vector product $A\mathbf{u}$ is defined? In which Euclidean space does $A\mathbf{u}$ lie?
The matrix-vector product is defined if $k=n$. In this case, $A\mathbf{u}$ is an element of $\mathbb{R}^m$.
5.(3+4+3+2) Consider the following linear homomorphism (keep an eye on the order of the vector coefficients in the image):

$$T : \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \to \begin{bmatrix} 2u_3 - u_1 + 2u_2 \\ u_1 + u_2 \\ 3u_1 - 5u_2 - 3u_3 \end{bmatrix}.$$

(a) Find the corresponding matrix A, such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

$$\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 0 \\ 3 & -5 & -3 \end{bmatrix}$$

(b) Find the kernel of T. (The answer is a set of vector(s)!)
The kernel is the set of all elements of the domain, that are mapped to the zero vector of the codomain, i.e. we need to find the solution set of the following homogeneous system:

$$\left[\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & -5 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 2 & 2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -7 & 0 \end{array}\right]$$

Performing backward substitution leads to $x_1 = x_2 = x_3 = 0$, the trivial solution. So the kernel of T is

$$\ker(T) = \{\mathbf{0} \in \mathbb{R}^3\}.$$

(c) Based on your answer in (b), is T injective? Justify your answer.
We know that T is injective, if and only if only the zero vector of the domain is mapped to the zero vector of the codomain. This is satisfied by (b), so T is indeed injective.

(d) Find the image of the vector

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

under T.

We calculate the image of the vector under T by computing the matrix vector product:

$$\begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & 0 \\ 3 & -5 & -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -8 \end{bmatrix}$$

The previous vector is the image of $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ under T.