Math 307 B,G – Exam 2

February 27, 2008

Name:____

Student ID:__________________________

Directions:

- You have 50 minutes to complete this exam.
- There are 5 problems, and the exam is worth 100 points.
- You are allowed a non-graphing calculator and a sheet of notes.
- Show all your work, and put a box around your $\boxed{\text{final answer}}$.
- **Note on units:** A kilogram (kg) is a unit of mass. A Newton (N) is a unit of force. One Newton is equal to one kg·m/s^2 (where m stands for meter and s stands for second).
- If you have any questions, ask me.

1	24	
2	20	
3	21	
4	20	
5	15	
Total	100	

1. I have a (rather magical) undamped oscillator consisting of a 0.5 kg mass hanging on a frictionless spring in a vacuum. The differential equation describing its position $u(t)$ is

$$0.5u'' + 2u = 0,$$

where $u(t)$ is measured in meters, and t is the time in seconds.

(**Note:** The different parts of this problem are independent except for the fact that they all refer to the above equation. You do not need any information from one part to do the other parts.)

(a) (2 points) What is the spring constant k? (Be sure to give units.)

Solution: Since the coefficient of u'' (i.e. 0.5) matches the numerical value of the mass in kg, the equation is in its standard form $mu'' + ku = 0$ (i.e. it has not been scaled), so the spring constant, in N/m, is the coefficient of u:

$$\boxed{k = 2 \text{ N/m} = 2 \text{ kg/s}^2}.$$

(Either of the above forms for the units is correct.)

(b) (10 points) If I push the mass 1 meter from equilibrium in the *negative* direction and then set it in motion with an initial velocity of 2 m/s in the *opposite* direction, find its position u as a function of time. Write the solution in the form $u(t) = R\cos(\omega_0 t - \delta)$, and determine the period, amplitude, and phase of the motion.

Solution: The characteristic equation is $0.5r^2 + 2 = 0$, with roots $r = \pm 2i$, so the natural angular frequency is $\omega_0 = 2$. Therefore, the general solution and its derivative are

$$u(t) = A\cos 2t + B\sin 2t$$
$$u'(t) = -2A\sin 2t + 2B\cos 2t.$$

The initial conditions are $u(0) = -1$ meter and $u'(0) = +2$ m/s. Therefore, setting $t = 0$ in the above equations gives

$$-1 = u(0) = A$$
$$2 = u'(0) = 2B,$$

so $A = -1$ and $B = 1$, and the **solution** is

$$\boxed{u(t) = -\cos 2t + \sin 2t}.$$

The **amplitude** is

$$R = \sqrt{A^2 + B^2} = \sqrt{1+1} = \boxed{\sqrt{2} \text{ meters}}.$$

Since the point $(A, B) = (-1, 1)$ is in the second quadrant, the **phase** is

$$\delta = \pi + \arctan(B/A) = \pi + \arctan(-1) = \pi - \pi/4 = \boxed{3\pi/4 \text{ radians}}.$$

Rewriting the solution in **amplitude-phase form** we have

$$\boxed{u(t) = \sqrt{2}\cos(2t - 3\pi/4)}.$$

The **period** is

$$T = 2\pi/\omega_0 = 2\pi/2 = \boxed{\pi \text{ seconds}}.$$

1. (continued) Recall that the equation for my undamped oscillator was

$$0.5u'' + 2u = 0.$$

(d) (4 points) To make my oscillator do more interesting things, I purchased a super-hi-tech force field generator that can exert a periodic force of $42\cos(\omega t)$ Newtons on the mass, and the generator has a knob that allows me to vary the angular frequency ω. For what value of ω will the system exhibit resonance?

Solution: Resonance occurs when the driving frequency ω equals the natural angular frequency $\omega_0 = \sqrt{k/m} = \sqrt{2/0.5} = \boxed{2 \text{ rad/s}}$.

(e) (8 points) I can also adjust my force field generator so that it does not produce a driving force, but instead exerts a damping force proportional to the mass's speed, with damping coefficient $\gamma = 1$ N·s/m. Write down the new differential equation describing the damped, non-forced system, and find its general solution. What are the quasi-frequency and quasi-period of the motion?

Differential equation: Since there is no external force, the differential equation governing the motion is $mu'' + \gamma u' + ku = 0$, or

$$\boxed{0.5u'' + u' + 2u = 0}.$$

General solution: The characteristic equation is $0.5r^2 + r + 2 = 0$, which has roots

$$r_1, r_2 = \frac{-1 \pm \sqrt{1^2 - 4(0.5)(2)}}{2(0.5)} = -1 \pm \sqrt{-3} = -1 \pm i\sqrt{3}.$$

Thus, the general solution is

$$\boxed{u(t) = c_1 e^{-t} \cos\left(\sqrt{3}\,t\right) + c_2 e^{-t} \sin\left(\sqrt{3}\,t\right)}.$$

Quasi-frequency: The quasi-frequency is the imaginary part of the roots (i.e. the coefficient of t inside the sine and cosine terms of the solution), so

$$\boxed{\mu = \sqrt{3} \text{ rad/s}}.$$

Quasi-period: The quasi-period is

$$T_d = \frac{2\pi}{\mu} = \boxed{\frac{2\pi}{\sqrt{3}} \text{ seconds}}.$$

2. Consider the following differential equation:

$$t^2y'' - 3ty' + 4y = 0; \quad t > 0.$$

(a) (5 points) Show that $y_1 = t^2$ is a solution to the above equation.

Solution: Simply plug $y_1 = t^2$ into the equation and show that we get 0:

$$\begin{aligned} t^2y_1'' - 3ty_1' + 4y_1 &= t^2(t^2)'' - 3t(t^2)' + 4t^2 \\ &= t^2(2) - 3t(2t) + 4t^2 \\ &= 2t^2 - 6t^2 + 4t^2 \\ &= 0. \end{aligned}$$

(b) (15 points) Find a second, linearly independent solution.

Solution: Use the method of reduction of order; that is, set $y_2(t) = y_1(t)v(t) = t^2v(t)$ and then solve for $v(t)$. First compute the first two derivatives of y_2:

$$\begin{aligned} y_2 &= t^2v \\ y_2' &= t^2v' + 2tv \\ y_2'' &= t^2v'' + 4tv' + 2v. \end{aligned}$$

Now plug y_2 into the original equation:

$$t^2y_2'' - 3ty_2' + 4y_2 = t^2(t^2v'' + 4tv' + 2v) - 3t(t^2v' + 2tv) + 4(t^2v) = 0.$$

After simplifying, you should end up with $t^4v'' + t^3v' = 0$, or

$$tv'' + v' = 0$$

This is a first order equation in the variable $w = v'$ which is both separable and linear, so we can use either technique to solve it.
Separable: With the substitution $w = v'$, the equation becomes $tw' + w = 0$. First separate variables: $dw/w = -dt/t$. Now integrate both sides: $\ln|w| = -\ln t + C$, so $w = A/t$ (where $A = \pm e^C$).
First order linear: First put it into standard form (i.e. divide by t to make the coefficient of w' equal 1): $w' + \frac{1}{t}w = 0$. Then the integrating factor is $\mu(t) = e^{\int(1/t)\,dt} = t$. Multiply by this and integrate; you should get $(tw)' = 0$, so $tw = A$ (remember that the integral of 0 is a constant), so $w = A/t$.
In either case, $w = A/t$, so $v(t) = \int w(t)\,dt = A\ln t + B$, so

$$\boxed{y_2(t) = y_1(t)v(t) = At^2\ln t + Bt^2}.$$

This is actually the general solution. Since I only asked for one solution that's not a constant multiple of t^2, you can also simply take $\boxed{y_2(t) = t^2\ln t}$.

3. For each of the following differential equations, write down the general form that you would use to find a solution using the method of undetermined coefficients. Include both the particular and homogeneous parts, but don't actually solve for the constants.

   (a) (7points) $y'' - 4y' + 4y = t^2 e^{2t}$

   Solution: The characteristic equation is $r^2 - 4r + 4 = (r-2)^2 = 0$, which has a repeated root $r = 2$, so the homogeneous (complementary) solution is

$$\boxed{y_c = c_1 e^{2t} + c_2 t e^{2t}}.$$

   For the particular solution, my first guess would be $(At^2 + Bt + C)e^{2t}$ (an arbitrary degree 2 polynomial times e^{2t}), but the terms Bte^{2t} and Ce^{2t} are both in the homogeneous solution, so this is insufficient. My second guess would be $t(At^2 + Bt + C)e^{2t}$, but the term Cte^{2t} is still part of the homogeneous solution, so I need to multiply by t again. Thus, the particular solution has the form

$$\boxed{y_p = t^2(At^2 + Bt + C)e^{2t}},$$

   and of course the whole solution is $\boxed{y = y_c + y_p}$.

   (b) (7 points) $y'' - 5y' + 4y = 6e^{4t} + 2t^3 + 1$

   Solution: The characteristic equation is $r^2 - 5r + 4 = (r-1)(r-4) = 0$, which has roots $r_1 = 1$, $r_2 = 4$, so the homogeneous solution is

$$\boxed{y_c = c_1 e^t + c_2 e^{4t}}.$$

   My first guess for a particular solution would be $Ae^{4t} + Bt^3 + Ct^2 + Dt + E$ (the first term because of the $6e^{4t}$, and the last four terms because of the $2t^3 + 1$). However, Ae^{4t} is part of y_c, so I need to multiply that term by t. Thus, the particular solution has the form

$$\boxed{Ate^{4t} + Bt^3 + Ct^2 + Dt + E},$$

   and the whole solution is $\boxed{y = y_c + y_p}$.

   (c) (7 points) $y'' + 3y' + 2 = te^{-2t}\sin(4t)$

   Solution: I actually intended for the above equation to be

$$y'' + 3y' + 2y = te^{-2t}\sin(4t), \tag{1}$$

   which is what most of you solved anyway (only a couple people noticed the missing y on the left-hand side). The characteristic equation of (1) is $r^2 + 3r + 2 = (r+1)(r+2) = 0$, which has roots $r_1 = -1$, $r_2 = -2$, so the homogeneous solution is

$$\boxed{y_c = c_1 e^{-t} + c_2 e^{-2t}}.$$

   The nonhomogeneous term is not part of y_c, so for the particular solution I would try

$$\boxed{y_p = (At + B)e^{-2t}\cos(4t) + (Ct + D)e^{-2t}\sin(4t)}.$$

   (Note that I need an arbitrary degree 1 polynomial in front of the cosine term and a *different* arbitrary degree 1 polynomial in front of the sine term.)

However, if you had been paying attention (which I apparently wasn't when I wrote the test, so I didn't take off points for solving the "wrong" version as above...), you would have realized that the given equation is the same as

$$y'' + 3y' = te^{-2t}\sin(4t) - 2. \tag{2}$$

The characteristic equation of (2) is $r^2 + 3r = r(r+3) = 0$, which has roots $r_1 = 0$, $r_2 = -3$, so the homogeneous solution is

$$\boxed{y_c = c_1 + c_2 e^{-3t}}.$$

(Note that $c_1 = c_1 e^{0t}$.) My first guess for the particular solution would be the same y_p as above, plus a constant term for the -2. However, since constants are part of y_c, I need to multiply the constant term by t, so the particular solution is

$$\boxed{y_p = (At + B)e^{-2t}\cos(4t) + (Ct + D)e^{-2t}\sin(4t) + Et}.$$

With either version, the whole solution is (as always) $\boxed{y = y_c + y_p}$.

4. The equation for a certain forced damped spring-mass system is

$$u'' + 5u' + 4u = 10\cos(3t),$$

where u is in feet, and t is in seconds.

(a) (5 points) Is the system underdamped, overdamped or critically damped? Justify your answer.

Solution: The discriminant is $\gamma^2 - 4km = 5^2 - 4(4)(1) = 9 > 0$, so the system is $\boxed{\text{overdamped}}$.

(b) (15 points) Find the steady state solution, and determine its amplitude and phase.

Solution: The steady state solution of a damped spring system is the part that does not die out, which is the particular solution arising from the driving force (as opposed to the homogeneous part of the solution, which is transient). According to the method of undetermined coefficients, the particular solution to the above equation has the form

$$U(t) = A\cos(3t) + B\sin(3t),$$

and we need to solve for the constants A and B. First take two derivatives of U:

$$\begin{aligned} U'(t) &= -3A\sin(3t) + 3B\cos(3t) \\ U''(t) &= -9A\cos(3t) - 9B\sin(3t). \end{aligned}$$

Now plug $U(t)$ into the differential equation:

$$\begin{aligned} U'' + 5U' + 4U = &\quad -9A\cos(3t) - 9B\sin(3t) \\ &+ 5\big[3B\cos(3t) - 3A\sin(3t)\big] \\ &+ 4\big[A\cos(3t) + B\sin(3t)\big] \quad = 10\cos(3t). \end{aligned}$$

Next, multiply the constants through on the left side, then group all the sine terms together and all the cosine terms together. After simplifying, you should end up with

$$(-5A + 15B)\cos(3t) + (-5B - 15A)\sin(3t) = 10\cos(3t).$$

Equating the coefficients of $\cos(3t)$ and $\sin(3t)$ on the two sides of the equation gives

$$\begin{aligned} -5A + 15B &= 10 \\ -5B - 15A &= 0. \end{aligned}$$

Solving, we get $A = -1/5$ and $B = 3/5$, so the **steady state solution** is

$$\boxed{U(t) = -\frac{1}{5}\cos(3t) + \frac{3}{5}\sin(3t)}.$$

The **amplitude** of the motion is

$$R = \sqrt{A^2 + B^2} = \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \boxed{\sqrt{\frac{2}{5}}\ \text{ft}}.$$

Since the point $(A, B) = (-\frac{1}{5}, \frac{3}{5})$ is in the second quadrant, the **phase** is

$$\delta = \pi + \arctan\left(\frac{3/5}{-1/5}\right) = \boxed{\pi - \arctan(3) \approx 1.8925 \text{ radians}}.$$

5. A mass of 2.5 kg is hung from a spring, stretching it 20 cm. The mass is in a viscous medium that exerts a force of 6 N when the speed of the mass is 4 m/s. The mass is lifted 8 cm above its equilibrium position and then thrown downward at 2 m/s. (Note: The acceleration of gravity is g = 9.8 m/s^2.)

(a) (12 points) Set up the initial value problem describing the motion of the mass. Specify which direction you want to be positive (you can pick either up or down to be positive, just tell me your choice). You do NOT need to solve the initial value problem.

Solution: First we need to identify the mass m, the damping coefficient γ, and the spring constant k:

$$m = \text{mass} = 2.5 \text{ kg}$$

$$\gamma = \frac{\text{damping force at 4 m/s}}{\text{4 m/s}} = \frac{6 \text{ N}}{4 \text{ m/s}} = 1.5 \text{ N}\cdot\text{s/m}$$

$$k = \frac{\text{weight of mass}}{\text{distance mass stretches spring}} = \frac{(2.5 \text{ kg})(9.8 \text{ m/s}^2)}{0.2 \text{ m}} = 122.5 \text{ N/m}$$

Therefore the differential equation for the position $u(t)$ is

$$\boxed{2.5u'' + 1.5u' + 122.5u = 0}.$$

If you decided that **up is positive**, then your initial conditions should be

$$\boxed{u(0) = 0.08 \text{ m}, \quad u'(0) = -2 \text{ m/s}}.$$

If you decided that **down is positive**, then your initial conditions should be

$$\boxed{u(0) = -0.08 \text{ m}, \quad u'(0) = 2 \text{ m/s}}.$$

(b) (3 points) Will the system return to its equilibrium position more than once? (Note: To answer this, you still do not need to solve the initial value problem.)

Solution: The system exhibits oscillations and returns to its equilibrium position more than once only if it is *underdamped*. The discriminant of the characteristic equation is

$$\gamma^2 - 4km = 1.5^2 - 4(122.5)(2.5) = -1222.75 < 0,$$

so the system *is* underdamped. Therefore the answer is $\boxed{\text{yes}}$.