Name:______________________________ **Student ID:**________________

Math 307 B,G | **Quiz 1** | **February 8, 2008**

1. Find the general solution to each of the following differential equations.

   (a) $2y'' - 3y' + y = 0$

   Characteristic equation: $2r^2 - 3r + 1 = (2r-1)(r-1) = 0$

   Roots: $r_1 = 1/2$, $r_2 = 1$ (distinct real)

   General solution: $y(t) = c_1 e^{t/2} + c_2 e^t$

   (b) $25y'' - 20y' + 4y = 0$

   Characteristic equation: $25r^2 - 20r + 4 = (5r-2)^2 = 0$

   Roots: $r_1 = r_2 = 2/5$ (repeated real)

   General solution: $y(t) = c_1 e^{2t/5} + c_2 t e^{2t/5}$ (Notice the factor of t in the second term!)

2. (a) Solve the initial value problem: $y'' + 6y' + 13y = 0$, $y(0) = 4$, $y(\pi/4) = 2$. (Note: The initial conditions both refer to y, not y'.)

   Characteristic equation: $r^2 + 6r + 13 = 0$

   Roots: $r_1, r_2 = \dfrac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2 \cdot 1} = \dfrac{-6}{2} \pm \dfrac{\sqrt{-16}}{2} = -3 \pm 2i$ (complex conjugates)

   General solution: $y(t) = c_1 e^{-3t}\cos(2t) + c_2 e^{-3t}\sin(2t)$

   Now use the initial conditions to solve for c_1 and c_2:

   $$\begin{aligned} 4 &= y(0) = c_1 \cdot 1 + c_2 \cdot 0 = c_1, \text{ and} \\ 2 &= y(\pi/4) = c_1\cos(\pi/2) + c_2 e^{-3\pi/4}\sin(\pi/2) = c_2 e^{-3\pi/4}, \end{aligned}$$

   so $c_1 = 4$ and $c_2 = 2e^{3\pi/4}$. Thus, the solution is

   $$y(t) = 4e^{-3t}\cos(2t) + 2e^{3\pi/4 - 3t}\sin(2t).$$

   (b) What is $\lim_{t\to\infty} y(t)$?

   The two terms in $y(t)$ each contain the exponential factor e^{-3t} multiplied by a bounded function ($c_1\cos(2t)$ and $c_2\sin(2t)$, respectively), so the limit is 0 since $\lim_{t\to\infty} e^{-3t} = 0$.

More questions on the back!!!

3. (a) Write the complex number $2e^{i\pi/4}$ in Cartesian form.

By Euler's formula,

$$2e^{i\pi/4} = 2(\cos(\pi/4) + i\sin(\pi/4)) = 2(\sqrt{2}/2 + i\sqrt{2}/2) = \sqrt{2} + i\sqrt{2}.$$

(b) Write the complex number $-4i$ in polar form.

We want to write $-4i = re^{i\theta}$, where $r = |-4i| = \sqrt{0^2 + (-4)^2} = 4$ and θ is the angle between the point $-4i$ and the positive x-axis (i.e. real axis), measured in the counterclockwise direction. Since $-4i$ lies on the negative y-axis (i.e. imaginary axis), we have $\theta = 3\pi/2$ (or any angle that differs from $3\pi/2$ by an integer multiple of 2π – for example, $\theta = -\pi/2$). Thus,

$$-4i = 4e^{3\pi i/2} = 4e^{-i\pi/2}.$$

4. Find the following antiderivatives. (Hint: These should be easy.)

(a) $$\int \frac{2xe^{x^2}}{e^{x^2}+5}dx = \int \frac{1}{u}du = \ln|u| + C = \ln\left(e^{x^2}+5\right) + \text{C}$$

(where $u = e^{x^2} + 5$, $du = 2xe^{x^2}dx$)

(b) $$\int \frac{1}{t}\cos(\ln t)dt = \int \cos(u)\,du = \sin(u) + C = \sin(\ln t) + C$$

(where $u = \ln t$, $du = \frac{1}{t}\,dt$)

(c) $$\int \frac{x^2 - \sin(3x)}{(x^3+\cos(3x))^2}dx = \frac{1}{3}\int u^{-2}du = -\frac{1}{3}u^{-1} + C = \frac{-1}{3(x^3+\cos(3x))} + C$$

(where $u = x^3 + \cos(3x)$, $du = [3x^2 - 3\sin(3x)]dx$)

(d) $$\int \frac{1}{t^3}e^{-1/t^2}dt = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{-1/t^2} + C$$

(where $u = -1/t^2$, $du = 2/t^3\,dt$)