Name:________________________________ **Student ID:**________________

Math 307 B,G

Quiz 4

March 7, 2008

1. Compute the Laplace transform of $f(t) = te^t$. Be sure to indicate where it exists, and to justify any limits you take.

   By definition,

$$\mathcal{L}\left\{te^t\right\}(s) = \int_0^\infty e^{-st} \cdot te^t \, dt = \int_0^\infty te^{(1-s)t} \, dt.$$

   To compute the integral, integrate by parts with $u = t$, $dv = e^{(1-s)t}dt$. Then

$$\begin{aligned}\int_0^\infty te^{(1-s)t} \, dt &= \left[\frac{1}{1-s}te^{(1-s)t} - \int \frac{1}{1-s}e^{(1-s)t} \, dt\right]_0^\infty \\ &= \left[\frac{1}{1-s}te^{(1-s)t} - \frac{1}{(1-s)^2}e^{(1-s)t}\right]_0^\infty \\ &= \lim_{t\to\infty}\left[\frac{1}{1-s}te^{(1-s)t} - \frac{1}{(1-s)^2}e^{(1-s)t}\right] - \left[0 - \frac{1}{(1-s)^2}\right].\end{aligned}$$

   Now, if $s > 1$, then $e^{(1-s)t} \to 0$ as $t \to \infty$, and applying L'Hopital's rule shows that $te^{(1-s)t} \to 0$ as well. Therefore, the limit as $t \to \infty$ of the entire expression inside the first set of brackets is 0 as long as $s > 1$, so the Laplace transform $\boxed{\text{exists for all } s > 1}$ and

$$\mathcal{L}\left\{te^t\right\}(s) = \int_0^\infty te^{(1-s)t} \, dt = \boxed{\frac{1}{(1-s)^2}}.$$

2. Find the inverse Laplace transform of $F(s) = \dfrac{3s+7}{s^2+2s+5}$.

   Since the denominator is an irreducible quadratic, we know the inverse transform will be some combination of exponentials times cosines and sines. Start by completing the square in the denominator, then manipulate the expression until it looks like some linear combination of forms that appear in the table:

$$F(s) = \frac{3s+7}{s^2+2s+1+4} = \frac{3(s+1)+4}{(s+1)^2+4} = 3\cdot\frac{s+1}{(s+1)^2+2^2} + 2\cdot\frac{2}{(s+1)^2+2^2}.$$

   (Notice that I need to have an $s+1$ in the numerator of the first term since $(s+1)^2$ appears in the denominator.) Therefore,

$$\mathcal{L}^{-1}\left\{F(s)\right\}(t) = 3\cdot\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+2^2}\right\} + 2\cdot\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\} = \boxed{3e^{-t}\cos 2t + 2e^{-t}\sin 2t}.$$

3. Let J be the operator defined by $J\left\{f(t)\right\}(x) = \int_0^x f(t)\,dt$. Show that J is linear.

   Just use the properties of integrals:

$$\begin{aligned}J\left\{af(t)+bg(t)\right\}(x) = \int_0^x [af(t)+bg(t)] \, dt &= \int_0^x af(t)\,dt + \int_0^x bg(t)\,dt \\ &= a\int_0^x f(t)\,dt + b\int_0^x g(t)\,dt = aJ\left\{f(t)\right\} + bJ\left\{g(t)\right\}.\end{aligned}$$

   Remark: Observe that J is the operator that takes a function $f(t)$ and gives you the antiderivative of f which is 0 at $x = 0$. (The way to read the above notation is that $J\left\{f(t)\right\}$ is a function F, and the formula defining F is $F(x) = \int_0^x f(t)\,dt$; hence $F'(x) = f(x)$ and $F(0) = 0$.)