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Math 307 B,G **Quiz 5** **March 14, 2008**

Solve the following initial value problem.

$$y'' + 4y' + 3y = f(t); \qquad y(0) = 0,\ y'(0) = 0; \qquad f(t) = \begin{cases} 0, & 0 \le t < 2 \\ 3, & 2 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

Solution: First we want to rewrite $f(t)$ in terms of step functions so that we can easily take its Laplace transform. Since $f(t) = 0$ for $t < 2$ and the points where the formula for f changes are at $t = 2$ and $t = 4$, we know we can write $f(t)$ in the form

$$f(t) = u_2(t)g(t) + u_4(t)h(t)$$

for some functions g and h. We can determine g and h by considering the intervals $[2, 4)$ and $[4, \infty)$ separately:

- If $2 \le t < 4$, then $u_2(t) = 1$ and $u_4(t) = 0$, so the formula $f(t) = u_2(t)g(t) + u_4(t)h(t)$ simplifies to $f(t) = g(t)$. Since $f(t) = 3$ on this interval we must have $g(t) = 3$.
- If $t \ge 4$, then $u_2(t) = 1$ and $u_4(t) = 1$, so we have $f(t) = g(t)+h(t) = 3+h(t)$. Since $f(t) = 0$ on this interval we must have $3 + h(t) = 0$, or $h(t) = -3$.

Therefore,

$$f(t) = 3u_2(t) - 3u_4(t).$$

Remark: *Note that the above method works in general for writing a multipart function in terms of step functions, so you would do well to understand how to apply it to any function given by a multipart rule...*

Now we take the Laplace transform of both sides of the equation and solve for $\mathcal{L}\{y\}$. For notational convenience I will set $\mathcal{L}\{y\} = Y$.

$$\begin{aligned}
\mathcal{L}\{y''\}(s) + 4\mathcal{L}\{y'\}(s) + 3\mathcal{L}\{y\}(s) &= \mathcal{L}\{f(t)\}(s) \\
\big[s^2Y(s) - sy(0) - y'(0)\big] + 4\big[sY(s) - y(0)\big] + 3Y(s) &= 3\mathcal{L}\{u_2(t)\}(s) - 3\mathcal{L}\{u_4(t)\}(s) \\
(s^2 + 4s + 3)Y(s) &= \frac{3e^{-2s}}{s} - \frac{3e^{-4s}}{s} \\
(s^2 + 4s + 3)Y(s) &= (e^{-2s} - e^{-4s})\frac{3}{s} \\
Y(s) &= (e^{-2s} - e^{-4s})\frac{3}{s(s^2 + 4s + 3)}.
\end{aligned}$$

To solve the original equation we now need to find $y = \mathcal{L}^{-1}\{Y(s)\}$.

Remark: *Notice that I have factored out the exponential terms e^{-2s} and e^{-4s}. In general, whenever you have an exponential term e^{-cs} and you're looking for the inverse Laplace transform, you want to first find the inverse transform of whatever the e^{-cs} is multiplied by, and then use the inverse translation property (Theorem 6.3.1).*

According to the above remark, we first want to find the inverse Laplace transform of

$$\frac{3}{s(s^2 + 4s + 3)} = \frac{3}{s(s + 1)(s + 3)}.$$

To do this, we use the method of partial fractions to find constants A, B, and C such that

$$\frac{3}{s(s+1)(s+3)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3},$$

or

$$3 = A(s+1)(s+3) + Bs(s+3) + Cs(s+1).$$

Setting $s = 0$, $s = -1$, and $s = -3$, respectively, we find that $A = 1$, $B = -3/2$, and $C = 1/2$. Therefore, since $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}(t) = e^{at}$, we have

$$\begin{aligned} Y(s) &= (e^{-2s} - e^{-4s})\left[\frac{1}{s} + \frac{-3/2}{s+1} + \frac{1/2}{s+3}\right] \\ &= (e^{-2s} - e^{-4s})\mathcal{L}\left\{1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right\}(s) \\ &= e^{-2s}\mathcal{L}\left\{1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right\}(s) - e^{-4s}\mathcal{L}\left\{1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right\}(s). \end{aligned}$$

Now, by the inverse translation property (Theorem 6.3.1 in the book),

$$e^{-2s}\mathcal{L}\left\{1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right\}(s) = \mathcal{L}\left\{u_2(t)\left[1 - \frac{3}{2}e^{-(t-2)} + \frac{1}{2}e^{-3(t-2)}\right]\right\}(s),$$

and

$$e^{-4s}\mathcal{L}\left\{1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}\right\}(s) = \mathcal{L}\left\{u_4(t)\left[1 - \frac{3}{2}e^{-(t-4)} + \frac{1}{2}e^{-3(t-4)}\right]\right\}(s),$$

so

$$Y = \mathcal{L}\left\{u_2(t)\left[1 - \frac{3}{2}e^{-(t-2)} + \frac{1}{2}e^{-3(t-2)}\right] - u_4(t)\left[1 - \frac{3}{2}e^{-(t-4)} + \frac{1}{2}e^{-3(t-4)}\right]\right\}.$$

Taking the inverse transform of both sides, the solution is

$$\boxed{y(t) = u_2(t)\left[1 - \frac{3}{2}e^{-(t-2)} + \frac{1}{2}e^{-3(t-2)}\right] - u_4(t)\left[1 - \frac{3}{2}e^{-(t-4)} + \frac{1}{2}e^{-3(t-4)}\right]}.$$