Homework 1 **(due Friday, June 24, 2005)** **Math 126A**

Note: The abbreviation "TN" refers to the Math 126 Taylor Notes, available as TaylorNotes.pdf on the Math 126 website `http://www.math.washington.edu/~m126`

1. Let $T_2(x) = 1 + x + \frac{x^2}{2}$ be the Taylor polynomial of degree two for $f(x) = e^x$, centered at $x = 0$ (see Example 1.1 in TN). Verify directly by taking their derivatives that $T_2(x)$ and $f(x)$ satisfy the three conditions $T_2(0) = f(0)$, $T_2'(0) = f'(0)$, and $T_2''(0) = f''(0)$.

2. Let $T_3(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6}) - \frac{1}{4}(x - \frac{5\pi}{6})^2 + \frac{\sqrt{3}}{12}(x - \frac{5\pi}{6})^3$ be the Taylor polynomial of degree three for $f(x) = \sin x$, centered at $x = 5\pi/6$ (see Example 1.2 in TN). Verify directly by taking their derivatives that $T_3(x)$ and $f(x)$ satisfy the four conditions
$$T_3(\frac{5\pi}{6}) = f(\frac{5\pi}{6}), \quad T_3'(\frac{5\pi}{6}) = f'(\frac{5\pi}{6}), \quad T_3''(\frac{5\pi}{6}) = f''(\frac{5\pi}{6}), \quad \text{and} \quad T_3'''(\frac{5\pi}{6}) = f'''(\frac{5\pi}{6}).$$

3. Let $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3$ be the Taylor polynomial of degree three for a function $f(x)$, centered at the point $x = a$.
   (a) Find $T_3'(x)$, $T_3''(x)$, and $T_3'''(x)$.
   (b) Evaluate $T_3(x)$, $T_3'(x)$, $T_3''(x)$, and $T_3'''(x)$ at $x = a$ to verify that
$$T_3(a) = f(a), \quad T_3'(a) = f'(a), \quad T_3''(a) = f''(a), \quad \text{and} \quad T_3'''(a) = f'''(a).$$

4. Find the Taylor polynomial of the given degree for the given function, centered at the given point. (Leave your answers as sums of powers of $(x-a)$, as in Example 1.2 in TN.)
   (a) $f(x) = \ln x$, degree 2, centered at $x = 1$.
   (b) $f(x) = \sqrt{1+x}$, degree 3, centered at $x = 0$.
   (c) $f(x) = \sin x$, degree 3, centered at $x = \pi/3$.
   (d) $f(x) = \cos x$, degree 4, centered at $x = \pi$.

5. Find the Taylor polynomial of degree n for $f(x) = \frac{1}{2-x}$, centered at $x = 0$.

6. Let $f(x) = \sqrt{1+x}$.

   (a) Find $|f^{(4)}(x)|$ and show that the function $g(x) = |f^{(4)}(x)|$ is decreasing for $x \geq 0$; conclude that
   $$\max_{x\geq 0} |f^{(4)}(x)| = |f^{(4)}(0)|.$$

   (b) Use Taylor's Inequality to give an upper bound for the absolute value of the difference $f(x) - T_3(x)$ between the function $f(x) = \sqrt{1+x}$ and its Taylor polynomial $T_3(x)$ of degree 3, centered at $x = 0$, for $0 \leq x \leq 0.1$.

   (c) You found $T_3(x)$ in problem 4(b). Use your calculator to find both $f(0.1) = \sqrt{1.1}$ and $T_3(0.1)$ to as many decimal digits as your calculator shows. How does $|f(0.1) - T_3(0.1)|$ compare with your answer to part (b)?

7. This problem refines Example 1.4 in TN. Let $T_1(x)$ and $T_2(x)$ be the Taylor polynomials of degrees one and two for $f(x) = \sin x$, centered at $x = 0$.

   (a) Show that $T_1(x) = T_2(x)$. (So in this case, the tangent line is also the Taylor polynomial of degree 2.)

   (b) Use Taylor's Inequality with $n = 2$ to give an even better (i.e., smaller) upper bound on $|\sin x - x|$ for $|x| \leq 0.1$ than is given in Example 1.4.

8. This problem continues the previous problem. Throughout this problem, $f(x) = \sin x$, and all Taylor polynomials are centered at $x = 0$.

   (a) Show that for n odd, $T_n(x) = T_{n+1}(x)$. (Hint: When n is odd, $f^{(n+1)}(0) = 0$.)

   (b) When n is odd, find an upper bound on $|\sin x - T_n(x)| = |\sin x - T_{n+1}(x)| = |R_{n+1}(x)|$ for $|x| \leq 0.1$ by using Taylor's Inequality for $n + 1$.

   (c) What is the smallest value of n for which you can guarantee that
   $$|\sin x - T_n(x)| \leq 10^{-10}$$
   whenever $|x| \leq 0.1$?
   (Hint: By part (a), the smallest n will be odd. Use your answer to part (b) to set up an equation [actually, an inequality] for n. Do not try to solve this inequality for n directly because n appears in both a factorial and an exponent. Instead, try guess and check: guess $n = 1$, $n = 3$, etc., and see what your answer to part (b) becomes.)

   (d) For the n you found in part (c), use your calculator to find both $\sin(0.1)$ (be sure your calculator is set on radians, not degrees) and $T_n(0.1)$ to as many decimal digits as your calculator shows. [**Remark:** Calculators use approximations similar to Taylor polynomials to compute trigonometric, exponential, and logarithmic functions.]

9. Determine whether the following improper integrals converge. For some of these, the easiest solution is to actually find the limit. For others, you will want to use the Comparison

Test to show that the improper integral converges absolutely.

(a) $\displaystyle\int_1^\infty \frac{1}{x}\, e^{-x}\, dx$

(b) $\displaystyle\int_0^\infty x\, e^{-x}\, dx$

(c) $\displaystyle\int_0^\infty x\, e^{-x} \sin(x^3)\, dx$

(d) $\displaystyle\int_2^\infty \frac{1}{x \ln x}\, dx$

(e) $\displaystyle\int_3^\infty \frac{1}{x^3 \ln x}\, dx$

(f) $\displaystyle\int_2^\infty \frac{\ln x}{x^3 + \sin^2 x}\, dx$

(g) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^3}\, dx$

(h) $\displaystyle\int_2^\infty \frac{\cos(e^x)}{x(\ln x)^3}\, dx$

10. A series $\displaystyle\sum_{k=0}^{\infty} a_k$ for which the ratio of successive terms is a nonzero constant

$$\frac{a_{k+1}}{a_k} = r \quad \text{for } k \geq 0$$

is called a *geometric series*. (Example 2.5 in TN is one example.)

(a) Let $a = a_0$. Show that for $k \geq 0$, $a_k = ar^k$.

[So the series is of the form $a + ar + ar^2 + ar^3 + \cdots$.]

(b) If $|r| < 1$, show that the series converges, and its limit is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} = \frac{(\text{first term})}{1 - (\text{common ratio})}.$$