Homework 2 **(due Friday, July 1, 2005)** **Math 126A**

Note: The abbreviation "TN" refers to the Math 126 Taylor Notes, available as TaylorNotes.pdf on the Math 126 website `http://www.math.washington.edu/~m126`

Here is the problem that was leftover from last week's assignment:

11. Determine whether the following infinite series converge. You may want to use the Comparison Test, the Integral Test, or perhaps both.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k + (1/2)^k}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{10^k}$

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{k^{5/2}}{k^{7/2} + 1}$

(d) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{k^2}{k^2 + 1}$

(e) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k + k^2}$

(f) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k + \sqrt{k}}$

(g) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{k}\,\sin(k\pi/4)}{k^2 + 4}$

And here are the problems that are new for this week. They cover the material from sections 3.1, 3.2, and 3.3 in TN.

1. [This problem shows that odd functions have only odd powers of x in their Maclaurin series, and even functions have only even powers of x.]

   (a) If $f(x)$ is an odd function (i.e, $f(-x) = -f(x)$), show that $f(0) = 0$.

   (b) If $f(x)$ is an odd function (i.e, $f(-x) = -f(x)$), use the chain rule to show that $f'(x)$ is an even function (i.e., $f'(-x) = f'(x)$).

   (c) If $f(x)$ is an even function (i.e, $f(-x) = f(x)$), use the chain rule to show that $f'(x)$ is an odd function (i.e., $f'(-x) = -f'(x)$).

   (d) Suppose $f(x)$ is an odd function. Use parts (b) and (c) to show that $f'(x)$ is even, $f''(x)$ is odd, $f'''(x)$ is even, etc., so $f^{(k)}(x)$ is an odd function for all even k. Use part (a) to conclude that $f^{(k)}(0) = 0$ for all even k, and thus the Maclaurin series $\displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for $f(x)$ has only odd powers of x. [For example, $\sin x$ is an odd function: $\sin(-x) = -\sin(x)$, and Example 3.3 in TN shows that its Maclaurin series has only odd powers of x.]

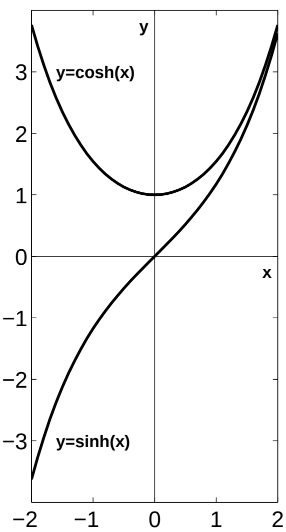
(e) Suppose $f(x)$ is an even function. Use parts (b) and (c) to show that $f'(x)$ is odd, $f''(x)$ is even, $f'''(x)$ is odd, etc., so $f^{(k)}(x)$ is an odd function for all odd k. Use part (a) to conclude that $f^{(k)}(0) = 0$ for all odd k, and thus the Maclaurin series $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$ for $f(x)$ has only even powers of x. [For example, $\cos x$ is an even function: $\cos(-x) = \cos(x)$, and Example 3.4 in TN shows that its Maclaurin series has only even powers of x.]

2. The *hyperbolic sine* function $\sinh(x)$ (pronounce "sinh" like "cinch") and the *hyperbolic cosine* function $\cosh(x)$ (pronounce "cosh" like the first syllable in "kosher") are defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}.$$

The following statements are easy to show, and you may use them in this problem without showing them:

- $\frac{d}{dx}(\sinh(x)) = \cosh(x)$,
- $\frac{d}{dx}(\cosh(x)) = \sinh(x)$,
- $\cosh(x) > 0$ for all x
  (so $\sinh(x)$ is always increasing),
- $\sinh(x) > 0$ for $x > 0$
  (so $\cosh(x)$ is increasing for $x > 0$),
- $\sinh(x) < 0$ for $x < 0$
  (so $\cosh(x)$ is decreasing for $x < 0$),



(a) Show that $\sinh(x)$ is an odd function and that $\cosh(x)$ is an even function.

(b) Find the Maclaurin series for $f(x) = \sinh(x)$ by evaluating $f(0)$, $f'(0)$, $f''(0)$, etc.

(c) Find the Maclaurin series for $f(x) = \cosh(x)$ by evaluating $f(0)$, $f'(0)$, $f''(0)$, etc.

(d) Show that the Maclaurin series for $\cosh(x)$ converges to $\cosh(x)$ for $x \geq 0$.
(Hint: Example 3.2 in TN for $x \geq 0$ is similar.)

[Note: The Maclaurin series for $\cosh(x)$ also converges to $\cosh(x)$ for $x < 0$, and the Maclaurin series for $\sinh(x)$ converges to $\sinh(x)$ for all x, but you are not asked to show these statements here as part of problem 2.]

3. Let $f(x) = \dfrac{1}{2-x}$. In problem 5 on Homework 1, you found the Taylor polynomial $T_n(x)$ of degree n for this function $f(x)$, centered at $x = 0$; this Taylor polynomial $T_n(x)$ is the n^{th} partial sum of the Maclaurin series for $f(x)$. In this problem, you will get the Maclaurin series for $f(x)$ in a different way.

   (a) Write $f(x)$ as follows:

$$\frac{1}{2-x} = \frac{1}{2(1-(x/2))} = \frac{1}{2} \cdot \frac{1}{1-(x/2)}.$$

   Now substitute $x/2$ in for u in $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots$ to find the Maclaurin series for $\dfrac{1}{1-(x/2)}$, and then multiply by $1/2$ to get the Maclaurin series for $f(x)$.

   (b) Find the radius of convergence R for the Maclaurin series for $f(x) = \dfrac{1}{2-x}$, and show that this series converges to $f(x)$ for $|x| < R$ and diverges for $|x| \geq R$.
   (Hint: Example 3.7 in TN is similar.)

4. Find the Maclaurin series for $f(x) = \ln(1+x)$ by evaluating $f(0)$, $f'(0)$, $f''(0)$, etc.
   [Your answer should agree with the answer to Example 3.10 in TN.]

5. Find the Maclaurin series for $\ln(1+x^2)$ by substituting x^2 in for x in your answer to problem 4.

6. (a) Substitute $-x$ in for u in the Maclaurin series $e^u = 1 + u + \dfrac{u^2}{2!} + \dfrac{u^3}{3!} + \dfrac{u^4}{4!} + \cdots$ to get the Maclaurin series for e^{-x}.

   (b) Find the Maclaurin series for $\sinh(x)$ by subtracting the Maclaurin series for e^{-x} from the Maclaurin series for e^x, and then dividing by 2.
   [You should get the same answer as in problem 2(b).]

   (c) Find the Maclaurin series for $\cosh(x)$ by adding the Maclaurin series for e^x to the Maclaurin series for e^{-x}, and then dividing by 2.
   [You should get the same answer as in problem 2(c).]

   (d) Differentiate the Maclaurin series for $\sinh(x)$ term by term and show that you get the Maclaurin series for $\cosh(x)$. (Hint: Example 3.9 in TN is similar.)

   (e) Differentiate the Maclaurin series for $\cosh(x)$ term by term and show that you get the Maclaurin series for $\sinh(x)$.

7. (a) Let $f(x)$ be a continuous function. Use the Fundamental Theorem of Calculus to show that $F(x) = \displaystyle\int_0^x f(t)\,dt$ is the antiderivative of $f(x)$ which also satisfies $F(0) = 0$.

(b) Let $F(x)$ be the antiderivative of $f(x) = e^{-x^2}$ which also satisfies $F(0) = 0$.

[Note: In Example 3.11 in TN, we found the Maclaurin series for $F(x)$ by taking the *indefinite integral* of the Maclaurin series for $f(x)$ term by term, and then solving for the constant C. In this problem, we will get the Maclaurin series for $F(x)$ in a different way.]

Substitute the Maclaurin series for $f(x)$ obtained in Example 3.11 in TN into the formula for $F(x)$ in part (a) and compute the *definite integral* term by term:

$$F(x) \;=\; \int_0^x e^{-t^2}\,dt \;=\; \int_0^x \left(\sum_{j=0}^{\infty}(-1)^j \frac{t^{2j}}{j!}\right) dt \;=\; \sum_{j=0}^{\infty}\left(\int_0^x (-1)^j \frac{t^{2j}}{j!}\,dt\right).$$

Finish the computation by evaluating the definite integrals to get the Maclaurin series for $F(x)$. [Your answer should agree with the Maclaurin series for $F(x)$ that was obtained in Example 3.11 in TN.]

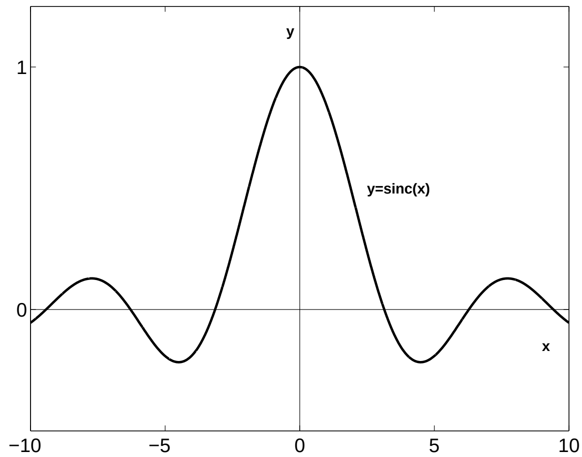
8. The *sinc* function (pronounce "sinc" like "sink") function is defined by

$$\operatorname{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

The sinc function is used frequently in electrical engineering and signal processing.

We know from Math 124 (or by L'Hôpital's Rule) that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, so the sinc function is continuous at $x = 0$. It turns out that the sinc function actually has derivatives of all orders, even at $x = 0$.

(a) Divide the Maclaurin series for $\sin x$ (Example 3.3 in TN) by x to obtain the Maclaurin series for $\operatorname{sinc}(x)$.



(b) The *sinc integral* function is defined by $\operatorname{Si}(x) = \int_0^x \operatorname{sinc}(t)\,dt$. Use either the method of Example 3.11 in TN or the method of problem 7 to find the Maclaurin series for $\operatorname{Si}(x)$.

9. (a) Show that $F(x) = \tan^{-1}(x)$ is the antiderivative of $f(x) = \dfrac{1}{1+x^2}$ which also satisfies $F(0) = 0$.

(b) Substitute $-x^2$ in for u in $\dfrac{1}{1-u} = 1 + u + u^2 + \cdots$ to find the Maclaurin series for $f(x) = \dfrac{1}{1+x^2}$.

(c) Find the radius of convergence R for the Maclaurin series for $f(x) = \frac{1}{1+x^2}$, and show that this series converges to $f(x)$ for $|x| < R$ and diverges for $|x| \geq R$.

(d) Use your answer to part (b) to find the Maclaurin series for $\tan^{-1}(x)$.

(e) Use the "fact" on page 28 of TN (the three bullets "•" near the top of the page) to determine the radius of convergence of the Maclaurin series for $\tan^{-1}(x)$.

10. Let $G(x)$ be the antiderivative of $\tan^{-1}(x)$ which also satisfies $G(0) = 0$. In this problem, we will compute the Maclaurin series for $G(x)$ in two ways.

(a) Use your answer to problem 9(d) to find the Maclaurin series for $G(x)$.

(b) Use integration by parts to find the indefinite integral $\int \tan^{-1}(x)\, dx$. Then evaluate the constant C in the indefinite integral to find $G(x)$, the antiderivative of $\tan^{-1}(x)$ which also satisfies $G(0) = 0$.

(c) Use the formula for $G(x)$ in part (b), the Maclaurin series for $\tan^{-1}(x)$ from problem 9(d), and the Maclaurin series for $\ln(1 + x^2)$ from problem 5 to obtain the Maclaurin series for $G(x)$. [Your answer should agree with your answer to part (a).]

11. Also do these problems from the Stewart text: Section 11.4: #20, 21, 27, 30, 32, 37, 38.