Hints and Answers for Midterm Review Worksheet Math 126A 7/21/05

1. (a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2+3}{5n^2-n}$ diverges.

   Why? Since $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{n^2+3}{5n^2-n} = \frac{1}{5} \neq 0$, the series diverges by the Test for Divergence.

   (b) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^{3/2}+1}{5n^3-n}$ converges.

   Why? Compare with a p-series to show absolute convergence:

$$\left|(-1)^n\frac{n^{3/2}+1}{5n^3-n}\right| = \frac{n^{3/2}+1}{5n^3-n} \leq \frac{n^{3/2}+n^{3/2}}{5n^3-n^3} = \frac{2n^{3/2}}{4n^3} = \frac{1}{2n^{3/2}}$$

   for all $n \geq 1$. Since $3/2 > 1$, the series $\sum_{n=1}^{\infty} n^{-3/2}$ converges, so the original series converges absolutely.

   (c) $\displaystyle\int_1^{\infty} x^{-\ln x}\,dx$ converges.

   Why? First recall that $\int_1^{\infty} x^{-p}\,dx$ converges for $p > 1$. Now note that $\ln x$ is greater than 1 if x is greater than e, so we can compare $x^{-\ln x}$ to a function of the form x^{-p} for some $p > 1$ if x is large enough. To make this concrete, let's choose $p = 2$. We know that $\ln x \geq 2$ when $x \geq e^2$, so $x^{-\ln x} \leq x^{-2}$ for all $x \geq e^2$. Therefore,

$$\int_{e^2}^{\infty} x^{-\ln x}\,dx \leq \int_{e^2}^{\infty} x^{-2}\,dx < \infty.$$

   Since the area between 1 and e^2 is also finite, this implies that $\int_1^{\infty} x^{-\ln x}\,dx < \infty$ as well, so the integral converges.

   (d) $\displaystyle\int_0^{\infty} \left[(\cos x)^2 \ln 2\right]^x\,dx$ converges.

   Why? Compare with an exponential function, i.e. a function of the form $f(x) = r^x$ for some $r > 0$. Note that

$$\left[(\cos x)^2 \ln 2\right]^x = |\cos x|^{2x}(\ln 2)^x \leq (\ln 2)^x$$

   since $|\cos x| \leq 1$. Now, $(\ln 2)^x$ is an exponential function with $r = \ln 2 < 1$. Since $r < 1$, the integral $\int_0^{\infty} r^x\,dx$ converges. (That is, the integral is finite. You can check this by computing the integral explicitly.) Therefore, the original integral also converges by the comparison test.

2. Hint: $g(5) = 3$, $g'(5) = \dfrac{1}{2}\cdot 3^{-1}$, $g''(5) = \dfrac{-1}{4}\cdot 3^{-3}$, $g'''(5) = \dfrac{3}{8}\cdot 3^{-5}$

   Answer: $T_3(x) = 3 + \dfrac{1}{6}(x-5) - \dfrac{1}{8\cdot 27}(x-5)^2 + \dfrac{1}{48\cdot 81}(x-5)^3$

3. (a) Hint: Write $f(x) = \frac{x}{2} \cdot \frac{1}{1 + x^3/2}$.

   Answer: $f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k+1}}{2^{k+1}}$. Radius of convergence is $r = 2^{1/3}$.

   (b) Answer: $\int f(x)\,dx = C + \sum_{k=0}^{\infty} (-1)^k \frac{x^{3k+2}}{2^{k+1}(3k+2)}$. The radius of convergence is the same as above.

4. (a) $2\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$

   (b) $\frac{|\mathbf{b}|}{|\mathbf{a}|}\,\mathbf{a}$

   (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (or any scalar multiple of this)

   (d) If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then

$$\begin{aligned}
(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) &= \mathbf{a} \times \mathbf{a} + \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{b} \\
&= \mathbf{0} - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} - \mathbf{0} \\
&= -2\mathbf{c}.
\end{aligned}$$

5. (a) Hint: The normal vectors to the planes are $\langle 1, 2, 2\rangle$ and $\langle 2, 0, 3, \rangle$. The line is perpendicular to both these vectors.

   Answer: $x = 6t + 1$, $y = t + 6$, $z = -4t + 10$

   (b) Hint: Use a new parameter (s, for example) to write down parametric equations for the second line.

   Answer: The lines intersect at $(-5, 5, 14)$.

6. (a) parallel

   (b) perpendicular

   (c) parallel

   (d) same

   (e) perpendicular

   (f) parallel

7. (a) The surface $z = x^2 + 4y^2$ is an example of a surface called an elliptic paraboloid. Its horizontal cross-sections are ellipses, and its vertical cross-sections are parabolas. See Example 4 in Section 12.6 for a picture of a rotated version of this surface.

   (b) Hint: If you set $z = 16$, you get the equation $x^2 + 4y^2 = 16$, or $\left(\frac{x}{4}\right)^2 + \left(\frac{y}{1}\right)^2 = 1$. This is the equation for an ellipse with semi-major axis 4 and semi-minor axis 1.
   Answer: One parameterization is $\mathbf{r}(t) = \langle 4\cos t, \sin t, 16\rangle$.

8. (a) $\text{comp}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{b}}\mathbf{a}$ if $\mathbf{a} \perp \mathbf{b}$ or if $|\mathbf{a}| = |\mathbf{b}|$. ($\perp$ means "is perpendicular to".)

(b) The scalar projections are different from just the dot products:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} \qquad \text{comp}_{\mathbf{b}}\mathbf{a} = \frac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|}$$

(c) $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$ only if $\mathbf{a} \perp \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$.

9. (a) N, (b) V, (c) N, (d) S, (e) S, (f) N

10. BORRRRINNNG!!

11. Hint: If r is the radius of convergence of the Taylor series, the given information implies that $3 \le r \le 4$.

    Answer: The series $\displaystyle\sum_{n=0}^{\infty}(-1)^n c_n$ converges, but there is not enough information to decide whether $\displaystyle\sum_{n=0}^{\infty}(-3)^n c_n$ converges.

12. (a) One possible triple is $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{i}+\mathbf{j}$, and $\mathbf{c} = 2\mathbf{i}+\mathbf{j}$. In this case, $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c} = \mathbf{k}$. In general, you can take $\mathbf{a}$ and $\mathbf{b}$ to be any two vectors, and then let $\mathbf{c} = \mathbf{b} + t\mathbf{a}$ for some scalar t. This gives all possible $\mathbf{c}$ such that $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$.

    (b) In two dimensions, the only vectors of length 5 orthogonal to $\mathbf{i}+2\mathbf{j}$ are $\sqrt{5}(-2\mathbf{i}+\mathbf{j})$ and $\sqrt{5}(2\mathbf{i}-\mathbf{j})$. In three dimensions, there are infinitely many such vectors because once you have one, you can rotate it through any angle in the plane orthogonal to $\mathbf{i}+2\mathbf{j}$ to get another one.

    To find all the possibilities, if $\mathbf{v} = \langle a,\, b,\, c\rangle$ is such a vector, then we must have $\mathbf{v}\cdot\langle 1,\, 2,\, 0\rangle = 0$ and $|\mathbf{v}| = 5$. Writing these equations in terms of the components of $\mathbf{v}$, we can solve for b and c in terms of a. This shows that $\mathbf{v}$ must have the form

$$\mathbf{v} = \left\langle\, a,\ -a/2,\ \pm\sqrt{25 - 5a^2/4}\,\right\rangle$$

    for some a in the interval $\left[-2\sqrt{5},\, 2\sqrt{5}\,\right]$.

13. (a) $x = t+2$, $y = -2t+5$, $z = 2t-1$.

    (b) $x = 2(t/b)+2$, $y = -4(t/b)+5$, $z = 4(t/b)-1$.