RECALL: An infinite set $X$ is countable iff there exists a bijection $f: X \to \mathbb{N}$.

**CANTOR HOTEL Handout:**

Think of the hotel rooms as the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

Think of the guests at the hotel as a set which is either countable or uncountable depending on whether there is a possible one-to-one room assignment of one guest per each room or not.

What the handout makes us prove:

1) \( X \text{ countable } \implies X \cup \{a\} \text{ is countable} \)
2) \( X \text{ countable } \implies X \cup \{a_1, a_2, \ldots, a_n\} \text{ is countable} \)
3) \( X, Y \text{ countable } \implies X \cup Y \text{ is countable} \)
4) \( X, Y \text{ countable } \implies X \times Y \text{ is countable} \)
   
   (alternatively: a union of countably many sets each of which is countable is countable)
5) The rationals are countable
6) The real numbers are not countable.
    
   In fact, not even \([0,1]\) is countable!

*You should know how to write a formal proof of any of these results.*

*The other results from Chapter 14, you should understand and be able to use them, but you don’t need to worry about their proofs*

(for the more ambitious, the proof of 14.3.3 is tough but beautiful!)

Some of the proofs:

2) \( X \text{ countable } \implies X \cup \{a_1, a_2, \ldots, a_n\} \text{ is countable} \) (union of disjoint sets)

**Handout idea:**

Ask everyone already in the hotel to move over by \( n \) rooms, (therefore freeing up the first \( n \) rooms.)

Give the new \( n \) guests the first \( n \) hotel rooms.

**Formal Proof:**

The set $X$ is countable if and only if there exists a bijection $f: X \to \mathbb{N}$.

Construct a bijection $g: X \cup \{a_1, a_2, \ldots, a_n\} \to \mathbb{N}$ as follows:

\[
g(a_i) = i \quad \text{for } 1 \leq i \leq n
\]

\[
g(x) = f(x) + n \quad \text{for all } x \in X
\]

Check this function maps $X \cup \{a_1, a_2, \ldots, a_n\}$ to $\mathbb{N}$ bijectively. Hence $X \cup \{a_1, a_2, \ldots, a_n\}$ is countable.
4) X, Y countable $\Rightarrow X \times Y$ is countable

Handout idea:

Denote the $n^{th}$ player on the $m^{th}$ team by $(m, n)$. Then all the players can be listed in an infinite matrix of the form:

\[
\begin{array}{ccc}
(1,1) & (1,2) & (1,3) \\
(2,1) & (2,2) & (2,3) \\
(3,1) & (3,2) & (3,3) \\
\vdots & \vdots & \vdots \\
\end{array}
\]

We can count the players across diagonals.

First note that:

- The $i^{th}$ diagonal has $i$ elements on it
- The $i^{th}$ diagonal consists of all elements $(x,y)$ with $x+y=i+1$

Hence:

- An element $(m,n)$ is on the $\underline{\text{____________________}}$ diagonal
- There are $1 + 2 + \cdots + (m - n - 2) = \frac{(m+n-2)(m+n-1)}{2}$ elements on the previous $m - n - 2$ diagonals and $(m, n)$ is the $n^{th}$ element in its own diagonal
- So the $(m,n)$ player gets the $\left(\frac{(m+n-2)(m+n-1)}{2} + n\right)^{th}$ key!

Formal proof: see text / do yourself
Infinite Cardinalities:

Recall:

- \(|X| = |Y| \iff \text{there exists a bijection } f: X \to Y\)

- \(\aleph_0 \equiv |\mathbb{N}|\) ("aleph null")
  Hence any infinite countable set has cardinality \(\aleph_0\) (\textit{why}?)

Any infinite set must have cardinality at least \(\aleph_0\). (\textit{why}?)

Are there infinite sets of larger cardinality? Yes. We saw that \(\mathbb{R}\) is not countable. So, \(|\mathbb{R}| > \aleph_0\).

- Theorem 14.3.3: For any set \(X\) (finite or infinite)
  \(|X| < |\mathcal{P}(X)|\)
  This proves that no largest cardinality exist! (\textit{why}?)

- Cantor showed: \(|\mathbb{R}| = |\mathcal{P}(\mathbb{N})| > \aleph_0\)

- Define \(\aleph_1\) to be the next smallest infinite cardinality larger than \(\aleph_0\)

\textit{Continuum hypothesis:} \(\aleph_1 = |\mathbb{R}|\)

This is an undecidable statement.
For simplicity, you may assume this as an axiom in your homework