Math 310 Spring 2008: Proofs By Induction Worksheet – Solutions

1. ***Prove that for all integers*** $n \geq 4$, $3^n \geq n^3$.

   Scratch work:

   (a) **What is the predicate P(n) that we aim to prove for all** $n \geq n_0$**?**

   $$P(n) : 3^n \geq n^3$$

   (b) **What is** $n_0 = ? = 4$

   (c) **So the base case consists of proving** $P(n_0)$**. Write out what this means, specifically for this problem.**

   $$P(4) : 3^4 \geq 4^3$$

   Now verify it.
   Since $3^4 = 81$ **and** $4^3 = 64$**, clearly** $P(4)$ **holds.**

   (d) **The induction step is to show that** $P(k) => P(k+1)$ **(for any** $k \geq n_0$**). Spell this out.**

   $$3^k \geq k^3 \Rightarrow 3^{k+1} \geq (k+1)^3 \text{ for any } k \geq 4$$

   i. **The *Induction Hypothesis* is** $P(k)$**. Write it out.**

   $$P(k) : 3^k \geq k^3$$

   ii. **Write out the goal:** $P(k+1)$**.**

   $$P(k+1) : 3^{k+1} \geq (k+1)^3$$

   iii. **Rewrite the LHS of** $P(k+1)$ **until you can relate it to the LHS of** $P(k)$**.**

   $$3^{k+1} = 3^k\dot{3} \geq 3\dot{k}^3$$

   iv. **Rewrite the RHS of** $P(k+1)$ **until you can relate it to the RHS of** $P(k)$**.**

   $$(k+1)^3 = k^3 + 3k^2 + 3k + 1.$$

   Want to show that this is less or equal to $3\dot{k}^3$

   v. **The induction hypothesis gives you the inequality between certain "chunks" of the RHS and LHS of** $P(k+1)$**. It remains to compare the remaining parts and show that the inequality holds between those too. Can you think of a way?**

   Use the back of the page to write a clear, correct, succint proof of the statement.

Prove that for all integers $n \geq 4$, $3^n \geq n^3$.

PROOF:

We'll denote by $P(n)$ the predicate $3^n \geq n^3$ and we'll prove that $P(n)$ holds for all $n \geq 4$ by induction in n.

1. Base Case $n = 4$: Since $3^4 = 81 \geq 64 = 4^3$, clearly $P(4)$ holds.

2. Induction Step: Suppose that $P(k)$ holds for some integer $k \geq 4$. That is, suppose that for that value of k, $3^k \geq k^3$. We want to show that $P(k+1) : 3^{k+1} \geq (k+1)^3$ must also hold.

Note that, by the induction hypothesis, $3^k \geq k^3$. Multiplying by positive k,

$$3^{k+1} = 3(3^k) \geq 3(k^3).$$

On the other hand, expanding $(k+1)^3$ we get $k^3 + 3k^2 + 3k + 1$. Hence it suffices to show that

$$3(k^3) \geq k^3 + 3k^2 + 3k + 1, \tag{1}$$

or, after subtracting k^3 from both sides, that $2(k^3) \geq 3k^2 + 3k + 1$. We'll show this piece-wise, comparing each of $3k^2$ and $3k + 1$ with k^3. For the first piece, for any $k \geq 3$, multiply by k^2 to get

$$k^3 \geq 3k^2. \tag{2}$$

For the second piece, note that for any integer $k \geq 4$[1],

$$k^3 \geq k^2 \geq 4k \geq 3k + 1. \tag{3}$$

Adding together inequalities (2) and (3),

$$k^3 + k^3 \geq 3k^2 + 3k + 1$$

which proves inequality (1), and hence it proves the induction step.

Since the statement holds for $n = 4$, and we have shown that if it holds for a certain integer $k \geq 4$ it must also hold for $k + 1$, the statement is true for all integers $n \geq 4$. QED

[1] Explicitly:
$k \geq 1 => 4k \geq 3k + 1$ (adding $3k$ to each side)
$k \geq 4 => k^2 \geq 4k$ (multiplying by positive k)
$k \geq 1 => k^3 \geq k^2$ (multiplying by positive k)
Combining all three, $k^3 \geq k^2 \geq 4k \geq 3k + 1$, for any $k \geq 4$.

2. Prove that 7 divides $2^{n+2} + 3^{2n+1}$ for any non-negative integer n.

Scratch work:

(a) **What is the predicate P(n) that we aim to prove for all $n \geq n_0$?**

$$P(n) : 7 \textbf{ divides } 2^{n+2} + 3^{2n+1}$$

(b) **What is** $n_0 =? = 0$

(c) **So the base case consists of proving $P(n_0)$. Write out what this means, specifically for this problem.**

$$P(0) : 7 \textbf{ divides } 2^{0+2} + 3^{2(0)+1}$$

Now verify it.
Since $2^{0+2} + 3^{2(0)+1} = 2^2 + 3 = 7$, certainly 7 divides it.

(d) **The induction step is to show that $P(k) => P(k+1)$ (for any $k \geq n_0$). Spell this out.**

If 7 divides $2^{k+2} + 3^{2k+1}$ for some $k \geq 0$, then it must also divide $2^{k+3} + 3^{2k+3}$

i. **The *Induction Hypothesis* is $P(k)$. Write it out.**
$$P(k) : 2^{k+2} + 3^{2k+1} = 7a \textbf{ for some integer } a$$

ii. **Write out the goal: $P(k+1)$.**
$$P(k+1) : 2^{k+3} + 3^{2k+3} = 7b \textbf{ for some integer } b$$

iii. **Rewrite the LHS of $P(k+1)$ until you can relate it to the LHS of $P(k)$.**

$2^{k+3}+3^{2k+3} = 2^{k+2}2+3^{2k+1}9 = 2^{k+2}2+3^{2k+1}(2+7) = 2(2^{k+2}+3^{2k+1})+3^{2k+1}7.$

iv. **Prove the induction step entirely.**

By induction hypothesis, $2^{k+2} + 3^{2k+1} = 7a$, so $2^{k+3} + 3^{2k+3} = 2(7a) + 3^{2k+1}7 = 7(2a + 3^{k+1})$.

Use the back of the page to write a clear, correct, succint proof of the statement.

Prove that 7 divides $2^{n+2} + 3^{2n+1}$ for any non-negative integer n.

PROOF:

We denote by $P(n)$ the predicate "7 divides $2^{n+2} + 3^{2n+1}$" and we'll use induction in n to show that $P(n)$ holds for all $n \geq 0$.

1. Base Case $n = 0$: Since $2^{0+2} + 3^{2(0)+1} = 2^2 + 3 = 7$ and 7 divides 7, $P(0)$ holds.

2. Induction Step: Suppose that $P(k)$ holds for some integer $k \geq 0$. That is, suppose that for that value of k, $2^{k+2} + 3^{2k+1} = 7a$ for some integer a. We want to show that $P(k+1)$ must also hold, i.e. that 7 must divide $2^{k+3} + 3^{2k+3}$.

Using the properties of the exponents and the distributivity and associativity of addition and multiplication,

$$2^{k+3} + 3^{2k+3} = 2^{k+2}2 + 3^{2k+1}9 = 2^{k+2}2 + 3^{2k+1}(2+7) = 2(2^{k+2} + 3^{2k+1}) + 3^{2k+1}7.$$

Hence, using the induction hypothesis,

$$2^{k+3} + 3^{2k+3} = 2(7a) + 3^{2k+1}7 = 7(2a + 3^{2k+1}).$$

This shows that 7 divides $2^{k+3} + 3^{2k+3}$, i.e. proves the induction step.

Since the statement holds for $n = 0$, and we have shown that if it holds for a certain integer $k \geq 0$ it must also hold for $k + 1$, the statement is true for all integers $n \geq 0$. QED