Math 310 : Some Review Problems for Midterm 1

1. ***Prove that if A, B and C are sets such that $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$***

2. ***For all real numbers x and y, prove that $|x+y| \leq |x| + |y|$.***

   You may use the definition of $|a|$, the addition, multiplication and transitivity laws for inequalities, and all basic arithmetic properties of real numbers.

3. ***Consider the following "proof"***
   THEOREM: For all sets A, B and C, if $C \subseteq A \cup B$ and $B \cap C = \emptyset$, then $C \subseteq A$.

   "PROOF": Let $A = \{a, b, c, d, e\}$ and $B = \{d, e, f, g\}$. If $C \subseteq A \cup B$, then the elements of C must be drawn from the list a, b, c, d, e, f, g. But $B \cap C = \emptyset$ so that B and C have no elements in common. Therefore, the elements of C must, in fact, be drawn from the list a, b, c. Since each of these elements is also an element of A, it follows that $C \subseteq A$.

   a) What is wrong with this argument?

   b) Write a correct proof of this result.

4. ***Prove that, for all non-negative integers n, $4^{2n+1} + 3^{n+2}$ is divisible by 13.***

5. ***Show that the sum of the squares of the first $n+1$ non-negative integers is*** $\dfrac{n(n+1)(2n+1)}{6}$.

6. ***Consider the sybolic statement $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, [(x \leq y) \Rightarrow x^2 \leq y^2]$.***

   a) ***Is the statement true or false? If true, prove it. If false, give a counterexample.***

   b) ***Write the symbolic negation of the statement.***

7. ***Draw up a truth table for the statement $(p \Rightarrow r) \wedge (r \Rightarrow q)$.***

8. ***Let propositions S, W, R and T be defined as follows:***

   S: The sun shines.
   W: The wind blows.
   R: The rain falls.
   T: The temperature rises.

   (i) Translate into English: $\neg(W \wedge R) \Leftrightarrow S$

   (ii) Translate into symbols: "The sun shines and the wind doesn't blow, and the temperature rises only if the rain falls."

   (iii) Suppose all of S, W, R, T are true. (Yes, it's a weird day:)) Decide which are true: a) $(S \Rightarrow W) \wedge (\neg R \wedge T)$ b) $(S \vee \neg R) \Leftrightarrow (T \vee \neg W)$ c) $\neg(R \vee \neg T) \wedge S$

9. ***Prove that for any sets A and B, $(A - B) \cap B = \emptyset$.***

10. ***Does the set $S = \{1 - 1/n \mid n \in \mathbb{Z}^+\}$ have a greatest element? Prove your answer.***

11. ***Let*** $f(x) = \sqrt{x+7}$**.**

a) Find its maximal domain X and list its codomain Y (in the real numbers).

b) With the domain and codomain from part a), is f injective? surjective? Prove your claims.

c) Write f as a composition of two functions g and h, none of which is the identity. Don't forget to specify the domain and the range of each.

d) Let

$$j(x) = \begin{cases} f(x), & \textbf{if } x \geq 2 \\ x^2 + c, & \textbf{if } x \leq 2. \end{cases}$$

For what values of $c \in \mathbb{R}$ is $j(x)$ a well-defined function?

12. ***Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two functions and define $f + g : \mathbb{R} \to \mathbb{R}$ by $(f+g)(x) = f(x) + g(x)$. If f and g are injective, is $f + g$ injective? If f and g are surjective, is $f + g$ surjective? If f is bijective, is $2f = f + f$ bijective? Prove your claims, or give counterexamples.***

13. ***Define "function". Define "contrapositive of a statement". Give examples of each.***

SOLUTION:

1. Wordy, more detailed proof:

Let A, B and C be sets such that $C \subseteq A$ and $C \subseteq B$. To prove $C \subseteq A \cap B$, we need to show that, for all x,

$$x \in C \Rightarrow x \in A \cap B.$$

So let x be an arbitrary element of C, $x \in C$. Since $C \subseteq A$, it follows that $x \in A$. Similarly, since $C \subseteq B$, it follows that $x \in B$. We have shown that $x \in A$ and $x \in B$, so it follows from the definition of intersection that $x \in A \cap B$.
Hence we have shown that $x \in C \Rightarrow x \in A \cap B$ and can therefore conclude that $C \subseteq A \cap B$. QED

2. Shorter symbolic proof:

Let A, B and C be sets such that $C \subseteq A$ and $C \subseteq B$.
Then $x \in C \Rightarrow x \in A$ (since $C \subseteq A$) and $x \in C \Rightarrow x \in B$ (since $C \subseteq A$).
Therefore $x \in C \Rightarrow (x \in A$ and $x \in B) \Rightarrow x \in A \cap B$ (from the definition of $A \cap B$).
Hence $C \subseteq A \cap B$. QED