1. p. 200, #33 How many Hamiltonian cycles does K_n have?

   The answer is $(n-1)!/2$. Indeed, every Hamiltonian cycle in K_n corresponds to a permutation of $[n]$, but since it is a cycle rather than a path, it doesn't matter where we start, that is, n permutations of the form

   $$i_1 i_2 \ldots i_n,\ i_2 i_3 \ldots i_n i_1,\ \ldots,\ i_n i_1 \ldots i_{n-1}$$

   all correspond to the same Hamiltonian cycle. The direction in which we traverse the cycle doesn't matter either, so that the permutations

   $$i_1 i_2 \ldots i_n \quad \text{and} \quad i_1 i_n i_{n-1} \ldots i_2$$

   define the same cycle. Thus the number of Hamiltonian cycles equals one $(2n)$th of the total number of permutations of $[n]$, that is $n!/2n = (n-1)!/2$.

2. p. 200, #34 Find the number of Hamiltonian cycles of $K_{m,n}$.

   We denote the vertices of $K_{n,m}$ by $1, \ldots n, 1', \ldots, m'$. If $C = v_1 \to v_2 \to \ldots \to v_{n+m} \to v_1$ is a Hamiltonian cycle in a bipartite graph and, say, v_1 belongs to its first part, then v_2 should belong to the second part, v_3 to the first, etc. In other words, all vertices with odd indices should belong to the first part while all vertices with even indices should belong to the second part, that is, both parts of the graph should contain exactly the same number of vertices. In other words, if $n \neq m$, then $K_{n,m}$ has zero Hamiltonian cycles.

   On the other hand, if $n = m$, then as in the previous problem, each Hamiltonian cycle corresponds to a string of the form $i_1 j_1' i_2 j_2' \ldots i_n j_n'$, where $i_1 \ldots i_n$ and $j_1 \ldots j_n$ are permutations of $[n]$. There are $(n!)^2$ pairs of such permutations. However, since in a cycle it doesn't matter what vertex do we start from and what direction do we take, we are counting each Hamiltonian cycle exactly $2n$ times: indeed the strings

   $$i_1 j_1' i_2 j_2' \ldots i_n j_n', \quad i_2 j_2' \ldots i_n j_n' i_1 j_1', \quad \ldots, \quad i_n j_n' i_1 j_1' i_2 j_2' \ldots i_{n-1} j_{n-1}'$$

   and

   $$i_1 j_n' i_n j_{n-1}' \ldots i_2 j_1, \quad i_2 j_1' i_1 j_n' \ldots i_3 j_2' \quad \ldots, \quad i_n j_{n-1}' i_{n-1} j_{n-2}' \ldots i_1 j_n'$$

   all correspond to the same Hamiltonian cycle. Thus there are $(n!)^2/2n = (n-1)!n!/2$ such cycles altogether.

3. p. 200, #41 Prove that the statement of the previous exercise is not true if we only assume that $d_x + d_z \geq n-1$.

   Consider graph G on n vertices labeled $1, 2, \ldots, n$ with edges (i, j) for all $1 \leq i < j \leq n-1$ and one additional edge $(1, n)$. Then $d_n = 1$, $d_1 = n-1$ and $d_i = n-2$ for all $1 < i < n$. In particular, G satisfies the condition $d_x + d_z \geq n-1$ for all vertices a and z. However, G doesn't contain a Hamiltonian cycle, since $d_n = 1$, and so there is no cycle in G that passes through n.

4. There are 100 towns in a country and some of them are connected by airlines. It is known that one can reach every town from any other (perhaps with several intermediate stops). Prove that you can fly around the country and visit ALL the towns making no more than 198 flights.

   Let us consider any spanning tree T of the graph of the airlines. We double each edge of this tree (there are 99 of them). The resulting multigraph $\widetilde{T}$ is connected, has 198 edges, and all of its vertices have even degree. Hence by Euler's theorem we can draw $\widetilde{T}$ with a pencil without lifting it from the paper. Such a drawing provides an order in which we can visit ALL the towns making exactly 198 flights.

5. Let G be a connected graph with at least two vertices. Prove that it has a vertex such that if this vertex is removed (along with all edges incident with it), the remaining graph is also connected.

   Since $G = (V, E)$ is connected, it has a spanning tree T. Let v be a leaf of T. (Such v exists since T is a tree.) Then $T \setminus v$ (that is, T with v removed) is a connected subgraph of $G \setminus v$ that uses all the vertices of $G \setminus v$. This implies that $G \setminus v$ is connected. Indeed, every two vertices of $G \setminus v$ can be connected by a path in $T \setminus v$, which is also a path in $G \setminus v$. The claim follows.

6. Prove that if a tree has a vertex of degree d, then it has at least d leaves.

   Denote the vertex of degree d by v and its d neighbors by $v_1, v_2, \ldots, v_d$. We start taking d walks from the vertex v, where in the i-th walk $(1 \le i \le d)$ we first go along the edge $\{v, v_i\}$. For each of these d walks we do the following: if v_i is a leaf, then we stay there; otherwise we move along any other edge incident to v_i to another vertex and so on. It is clear that in any of our d walks we cannot come to a vertex we have visited before in the same walk as this would mean existence of a cycle. Moreover, no two of our d walks can share a common vertex (except v) — this would also mean existence of a cycle. On the other hand, since our graph has a finite number of vertices, all our d walks must end somewhere. But the vertex a walk ends in must a leaf. Thus our d walks will end in d leaves any two of which are distinct, and the statement follows.