Reading assignment: Read carefully §10.1 and and p. 243 of §11.2. Also make sure you understand how Prüfer codes work (see solution to Problem #(5) on p. 228) — you will need this material for most of the problems on this homework.
Start reading §11.3.

Written assignments:

1. Let T be a tree on n vertices such that the degree of each vertex is either 1 or 3.
   (a) Show that n is even.
   (b) Determine how many of the vertices have degree 1 and how many have degree 3.
   (c) Compute the number of such labeled trees T.

2. Let $C(n, k)$ denote the number of labeled trees on n vertices in which a given vertex, say nth, has degree k. Show that
$$C(n,k) = \binom{n-2}{k-1}(n-1)^{n-k-1}.$$
Hint: How many times does n appear in the Prüfer code of such a tree?

3. Prove that the number of labeled trees on the vertex set $V = \{1, 2, \ldots, n\}$ that contain edge $\{1, 2\}$ is $2n^{n-3}$.
Hint: notice that every pair $\{i, j\} \subset V$ appears as an edge in exactly the same number of trees. Use this observation to determine which fraction of all the labeled trees contains a given pair as an edge.

4. p. 261, #(16)
Hint: use divisibility by 3.

Theorems that may be useful for this homework:

Cayley's theorem: The number of labeled trees on n vertices is n^{n-2}.

A refinement of Cayley's theorem: The number of labeled trees on n vertices $\{1, \ldots, n\}$ with the property that the degree of the i-th vertex is d_i (for all $i = 1, \ldots, n$) is given by
$$\binom{n-2}{d_0-1, \ldots, d_{n-1}-1} = \frac{(n-2)!}{(d_0-1)!(d_1-1)!\ldots(d_{n-1}-1)!}.$$

A theorem that implies both previous statements: There is a bijection between the set of labeled trees on the vertex set $[n]$ and the set of words of length $n-2$ written in the alphabet $[n]$. Under this bijection a tree whose degree sequence is $(d_1, \ldots, d_n)$ corresponds to a word that consists of $(d_1 - 1)$ 1's, $(d_2 - 1)$ 2's, etc.