1. ("Warm up")
   (a) How many strings can be written using exactly five letters A, three letters B, and no more than two letters C (and no other letters)?
   (b) We must choose a 5-member team from 12 girls and 10 boys. How many ways are there to make the choice so that there are no more than 3 boys on the team?
   (c) How many of graphs on 12 vertices labeled $1, 2, \ldots, 12$ have exactly fifteen edges?
   (d) Let G be a connected planar graph with 100 vertices and 140 edges. How many regions are in a plane drawing of G?
   (e) Let F be a forest with 100 vertices and 90 edges. How many new edges must be added without adding vertices to obtain a tree?
   (f) Suppose that a graph G has 1000 vertices and 3000 edges. Can G be planar?
   (g) Does there exist a graph on 7 vertices whose degree sequence equals $(5, 5, 4, 4, 3, 2, 2)$?
   (h) How many distinct perfect matchings does a complete bipartite graph $K_{9,9}$ have? (Recall that a matching is a subset of edges. Two matchings are distinct if they are not equal as sets.)

2. Prove by induction that $F_n < 2^n$ for all $n \geq 1$, where F_n denotes the n-th Fibonacci number. (Recall that $F_0 = F_1 = 1$.)

3. (a) How many solutions does the equation $x_1 + x_2 + x_3 + x_4 = 15$ have if all x_i's are integers satisfying $x_1 \geq 2$, $x_2 \geq -1$, $x_3 \geq 1$, and $x_4 \geq 4$?
   (b) Determine the number of solutions of the equation $x + y + z + t = 31$ if x, y, z and t are integers satisfying $1 \leq x \leq 9$, $1 \leq y \leq 9$, $1 \leq z \leq 9$, and $1 \leq t \leq 9$.

4. In a room there are 10 people, none of whom are older than 100 (ages are given in whole numbers only) but each of whom is at least 1 year old. Prove that one can always find two groups of people (possibly intersecting, but different) the sums of whose ages are the same. (You can also try to prove a stronger statement that there are in fact two disjoint sets of people that have the same sum of ages.)
   Hint: Use the Pigeonhole Principle.

5. (a) Let G be a connected graph with n vertices and n edges. How many cycles does G have? Explain!
   (b) Which trees are Eulerian graphs (i.e., graphs that have an Eulerian walk)? Explain!
   (c) Which labeled trees have Prüfer code that contain only one value? Explain!

6. Let T be a (finite) tree in which there are no vertices of degree two. Prove that at least half of the vertices of T are leaves.

7. Determine how many among all the labeled trees on 8 vertices have two vertices of degree 3, two vertices of degree 2, and four vertices of degree 1.

8. Is there a disconnected graph with degree sequence (4, 4, 3, 3, 3, 3, 3, 3)?
   Hint: what is the minimal size of the connected component of the vertex of degree d?

9. Consider $n \times n$ chessboard with some of the squares forbidden. Prove that if there exists a positive integer p such that each row and each column contains exactly p allowed squares, then it is possible to place n non-attacking rooks on this board.

10. Prove that every planar graph with fewer than 12 vertices has a vertex of degree ≤ 4.

11. Review problems from the midterm.