This week we will discuss bipartite graphs and matchings in bipartite graphs. The most important theorem we'll learn is Philip Hall's theorem (also known as Hall's marriage theorem). Please read all the lecture notes; see also $\S \S 10.1-10.3$ of "Discrete Mathematics. Elementary and Beyond."

## Written Assignments Due Wednesday, 1/13/17.

1. Is there a bipartite graph with degree sequence $(3,3,3,3,3,3,3,3,3,5,6,6,6)$ ?

Hint: use divisibility by 3 .
2. Let $G$ be a bipartite graph all of whose vertices have the same degree $d$. Show that there are at least $d$ distinct perfect matchings in $G$. (Two perfect matchings $M_{1}$ and $M_{2}$ are distinct if $M_{1} \neq M_{2}$ as sets.)
Hint: by one of the theorems proved in class, there is at least one matching M. Mentally delete the edges of $M$ from $G$.
3. Let $G=(X, Y)$ be a bipartite graph with $|X|=5$ and $|Y|=7$. Assume that every $x \in X$ has degree at least 4 and every $y \in Y$ has degree at least 1 . Show that there exists a perfect matching of $X$ into $Y$.
4. There are $N$ men and a group of women. Assume that for every $2 \leq k \leq N$, every $k$ men are acquainted with at least $k-2$ women. Show that it is possible to match $N-2$ (or more) men to the women they know.
Hint: bring two additional women that are acquainted with every one of the $N$ men.
5. (a) Let $G=(X, Y)$ be a bipartite graph with the property that the smallest degree of a vertex in $X$ is at least as large as the largest degree of a vertex in $Y$. Prove that there is a perfect matching of $X$ into $Y$.
Hint: mimic the proof with "sending letters" we saw in class.
(b) Fix two positive integers $k$ and $n$ so that $k<n / 2$. Let $G=(X, Y)$ be the bipartite graph in which the vertices of $X$ are the $k$-element subsets of $[n]$, and the vertices of $Y$ are the $(k+1)$-element subsets of $[n]$, and there is an edge between $x \in X$ and $y \in Y$ if and only if $x \subset y$. (For instance, $x^{\prime}=\{2,3, \ldots, k+1\}$ is an element of $X$ while $y^{\prime}=\{1,2, \ldots, k, k+1\}$ and $y^{\prime \prime}=\{1,2,3, \ldots, k, k+2\}$ are elements of $Y$; furthermore, there is an edge between $x^{\prime}$ and $y^{\prime}$, but there is no edge between $x^{\prime}$ and $y^{\prime \prime}$.) Prove that there is a perfect matching of $X$ into $Y$.
6. Consider the set $[m n]=\{1,2, \ldots, m n\}$. We partition $[m n]$ into $m$ sets $A_{1}, \ldots, A_{m}$ of size $n$. Let a second partitioning of $[m n]$ into $m$ sets of size $n$ be $B_{1}, \ldots, B_{m}$. Show that the sets $B_{i}$ can be renumbered in such a way that $A_{i} \cap B_{i} \neq \emptyset$ for all $i$.
Hint: Construct a bipartite graph on the vertex set $A_{1}, \ldots, A_{m}, B_{1}, \ldots, B_{m}$ by joining $A_{i}$ to $B_{j}$ by an edge if and only if $A_{i} \cap B_{j} \neq \emptyset$. Check that the condition of the Marriage theorem is satisfied.

