We continue discussing the probabilistic method. Please read the lecture notes and §10.2 (but skip the subsection "Independent events") and §§10.3, 10.4 in the book.

Written Assignments Due Friday, 2/24/17.

1. Let Ω be the set of all strings of length n in the alphabet $\{1, 2, 3\}$. We make Ω into a probability space by assuming that all strings are equally likely.
   (a) What is the expected number of 1s in a random string?
   (b) What is the expected number of '11' (i.e., two consecutive 1's)?
2. We say that a permutation π of $\{1, 2, \ldots, n\}$ transposes i and j if $\pi(i) = j$ and $\pi(j) = i$. For instance, if $n = 7$ and $\pi = (5, 3, 2, 6, 1, 7, 4)$, then π transposes 1 and 5; it also transposes 2 and 3. What is the expected number of transpositions in a random permutation of $\{1, 2, \ldots, n\}$? (We assume that all permutations of $\{1, 2, \ldots, n\}$ are equally likely.)
   Hint: for each pair $\{i, j\} \subseteq [n]$, define the corresponding indicator variable.
3. What is the expected number of cliques of size 4 in a randomly selected graph on the vertex set $[n]$? (We assume that all graphs on n vertices are equally likely.)
   Hint: for each quadruple of vertices, define the corresponding indicator variable.
4. We are given a list $L = (L_1, L_2, \ldots, L_k)$ of ordered triples $L_i = (a_i, b_i, c_i)$, so that for any i, the numbers a_i, b_i, and c_i are distinct elements of $[n]$. (The triples are not necessarily disjoint.)

   Let $p = p_1 p_2 \cdots p_n$ be a permutation of $\{1, 2, \ldots, n\}$. We say that p satisfies L_i if the entry b_i is between a_i and c_i in p. (It doesn't matter whether the order of these 3 entries in p is $a_i b_i c_i$ or $c_i b_i a_i$.)

   Prove that there exists a permutation p of $\{1, 2, \ldots, n\}$ that satisfies at least one third of all L_i, in any given list L.
   Hint: for a fixed list L, compute the expected number of L_i's that are satisfied by a random permutation.