Info about the Final Exam: The final exam will take place on Wednesday, March 15, 2:30-4:20, in our usual room. The final will be comprehensive. Make sure to re-read the lecture notes as well as the appropriate portions of the book.

You are allowed to bring in a simple scientific calculator and a hand-written, two-sided sheet of paper (format A4) with notes.

We will have a review session on Tuesday, March 14, 10-12 in Padelford C-36.

Practice problems: Make sure to review all the problems from the homeworks and the midterm. Below are some additional problems for you to practice. NOTE: the problems below touch only on some (but not all) of the topics we discussed this quarter. The first six problems are review problems while the last six problems would have been assigned as Homework #8 if this were not the last week of the quarter.

1. An $n \times n$ matrix is called a **permutation matrix** if all entries are 0 or 1, and there is precisely one 1 in each row and each column. Show that if M is an $n \times n$ matrix with all entries 0 or 1, and with exactly m 1s in each row and each column, then M can be written as a sum of permutation matrices.
   Hint: this is a matching problem.

2. Let $\{A_1, A_2, \ldots, A_n\}$ be a family of subsets of $\{1, 2, \ldots, n\}$. Consider the incidence matrix M of this family: M is an $n \times n$-matrix with $M_{ij} = 1$ if $j \in A_i$ and $M_{ij} = 0$ otherwise. Prove that if M is invertible, then the family possesses a system of distinct representatives.

3. (a) Determine the Ramsey number $r(10, 2)$.
   (b) Define $r(k, l, p)$ to be the smallest integer such that any coloring the edges of $K_{r(k,l,p)}$ (i.e., the complete graph with $r(k, l, p)$ vertices) with red, blue or green will result in a red k-clique, blue l-clique, or green p-clique. Prove that $r(3, 3, 3) \leq 17$.

4. **Using the probabilistic method**, show that for $n \geq 2$, it is possible to color the edges of K_n with n colors in such a way that there are no more than $\frac{n}{6}$ monochromatic triangles.

5. Recall that the Fibonacci recurrence is $f_n = f_{n-1} + f_{n-2}$ where $f_0 = f_1 = 1$. Show that the ordinary generating function of the Fibonacci sequence is $f(x) = \frac{1}{1-x-x^2}$.

6. Set up the generating function for the sequence $\{h_n\}_{n \geq 0}$, where h_n counts the number of weak compositions of n into 4 parts such that the first part does not exceed 2, the second part is an even number, the third part is an odd number that does not exceed 7, and the last part is at least one.

7. (a) Determine the exponential generating function for the sequence of factorials:
   $$(0!, 1!, 2!, \ldots, n!, \ldots).$$
   (b) Determine the exponential generating function for the sequence of plus and minus ones: $(1, -1, 1, -1, \ldots)$.

(c) Let α be a real number. Let the sequence $\{h_n\}_{n=0}^{\infty}$ be defined by $h_0 = 1$, and $h_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$ for $n \geq 1$. Determine the exponential generating function for this sequence.

8. There are $2n$ points marked on a circle. We want to divide them into pairs and connect the points in each pair by a segment (chord) in such a way that these segments do not intersect. Show that the number of ways to do that is equal to the nth Catalan number.

9. (a) Check that if $a(x)$ is the ordinary generating function of a sequence $(a_0, a_1, a_2, \ldots)$, then $\frac{1}{1-x} \cdot a(x)$ is the ordinary generating function of the sequence of partial sums $(a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)$.

   (b) Let $\{a_n\}_{n\geq 0}$ and $\{b_n\}_{n\geq 0}$ be two sequences, and let $A(x)$ and $B(x)$ be their respective exponential generating functions. Let us assume that we know that $B(x) = \frac{1}{1-x} \cdot A(x)$. What is the relationship between the two sequences?
   Hint: try to use part (a).

10. Let $a_0 = a_1 = 1$, and let $a_n = na_{n-1} + n(n-1)a_{n-2}$ for $n \geq 2$. Find the exponential generating function of the sequence $(a_0, a_1, \ldots, a_n, \ldots)$.

11. Let $a_0 = 0$, and let $a_{n+1} = (n+1)a_n + n!$ for $n \geq 0$. Find an explicit formula for a_n. (**Hint:** using results of Problem 9 might save you some computations.)

12. A town of n citizens has m red clubs $R_1, \ldots, R_m$ and m blue clubs $B_1, \ldots, B_m$. It is known that $|R_i \cap B_i|$ is odd for every i and that $|R_i \cap B_j|$ is even for every $i \neq j$. Prove that $m \leq n$.
Hint: Try to mimic the proof from class, but this time consider two incidence matrices: one corresponding to the red clubs and the citizens, and another one corresponding to the blue clubs and the citizens.