1. Show that for every prime p and $q = p^k$ there exists a projective plane of order q.
   Hint: consider $\mathbb{F}_q^3$ and its 1-dimensional and 2-dimensional subspaces.

2. Let $1 \le k \le n/2$ and let $\mathcal{F} = \{F_1, \ldots, F_m\} \subset 2^{[n]}$ be a Sperner family on $[n]$ all of whose members have size at most k. Show that $|\mathcal{F}| \le \binom{n}{k}$.

3. Prove the following extension of the LYM inequality. Let $\mathcal{F} \subset 2^{[n]}$ be a set system not containing $s+1$ nested sets: $F_1 \subset F_2 \subset \ldots \subset F_{s+1}$, and write f_k to denote the size of $\mathcal{F} \cap \binom{[n]}{k}$. Prove that
$$\sum_{k=1}^{n} f_k \binom{n}{k}^{-1} \le s.$$

4. (a) Let $\{A_1, \ldots, A_s\}$ be a collection of l-subsets of $[n]$, where $l < n/2$. Show that there exist s distinct $(l+1)$-subsets $B_1, \ldots B_s$ of $[n]$ such that $A_i \subset B_i$ for every $1 \le i \le s$.
   Hint: use P. Hall's Marriage theorem to match the collection $\{A_1, \ldots, A_s\}$ to the collection of $(l+1)$-subsets of $[n]$ that contain one or more of the sets A_i, $1 \le i \le s$.

   (b) Prove the following extension of the Erdös-Ko-Rado theorem. Let $1 \le k \le n/2$ and let $\mathcal{F}$ be a family of subsets of $[n]$ all of whose members have size at most k. Show that if $\mathcal{F}$ is both **intersecting** and **Sperner** then $|\mathcal{F}| \le \binom{n-1}{k-1}$.
   Hint: Use part (a) to reduce this claim to the Erdös-Ko-Rado theorem.

   (c) Let $\mathcal{F}$ be a family of subsets of $[n]$ which is both **intersecting** and **Sperner**. Assume also that $A \cup B \ne [n]$ for all $A, B \in \mathcal{F}$. Prove that $|\mathcal{F}| \le \binom{n-1}{\lfloor n/2 \rfloor - 1}$.
   Hint: Show that large sets can be replaced by their complements; apply part (b).