Math 582C **Homework #2** **January 26, 2009**

(1) Let $K \subset \mathbb{R}^d$ be a compact and convex set, and let $p \in \mathbb{R}^d$ be a given point. Show that there exists a unique nearest point $q \in K$ to p (that is, $||p-q|| < ||p-q'||$) for all $q' \in K - \{q\}$).

(2) (Gauss-Lucas Theorem) Let $f(z)$ be a non-constant polynomial in one complex variable z and let $z_1, \ldots, z_m$ be the roots of f (that is, the set of all solutions to the equation $f(z) = 0$). We interpret a complex number $z = x + iy$ as a point $(x, y) \in \mathbb{R}^2$. Prove that each root of the derivative $f'(z)$ lies in the convex hull $\mathrm{conv}(z_1, \ldots, z_m)$.
Hint: without loss of generality we may assume that $f(z) = (z - z_1) \cdots (z - z_m)$. If w is a root of $f'(z)$, then $\sum_{l=1}^{m} \prod_{j \neq l} (w - z_j) = 0$, and therefore, $\sum_{l=1}^{m} \prod_{j \neq l} \overline{(w - z_j)} = 0$. Multiply both sides of this last identity by $(w - z_1) \cdots (w - z_m)$ and express w as a convex combination of $z_1, \ldots, z_m$.

(3) Show that the convex hull of a compact set is compact.
Hint: One possible solution is to use Carathéodory's theorem to express $\mathrm{conv}\,(K)$ as a continuous image of a compact set.

(4) (a) Here is an application of Radon's theorem to combinatorics: Let $F_1, \ldots, F_m$ be subsets of $[n] := \{1, 2, \ldots, n\}$. Show that if $m \geq n + 2$ then there exist two disjoint nonempty sets $I \subset [m]$ and $J \subset [m]$ such that $\cup_{i \in I} F_i = \cup_{j \in J} F_j$.
Hint: Associate with each set F_i its incidence vector $v_i \in \mathbb{R}^n$. Use Radon's theorem.
(b) State and prove a similar application of Tverberg's theorem.

(5) Let X be an n-point set in $\mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is called a centerpoint of X if each closed half-space containing x contains at least $n/(d+1)$ points of X. Use Tverberg's theorem to show that each finite point set in $\mathbb{R}^d$ has at least one centerpoint.