Math 582C **Homework #3** **February 9, 2009**

(1) Show that if P is a 3-dimensional simple polytope, then $f_1(P) = \frac{3}{2}f_0(P)$ and $f_2(P) = 2 + \frac{1}{2}f_0(P)$, where f_0 is the number of vertices, f_1 is the number of edges and f_2 is the number of facets.
Hint: You can use the Euler formula: $f_0 - f_1 + f_2 = 2$.

(2) Let P be the d-dimensional cube. Show that if $\mathcal{O}$ is an acyclic orientation of $G(P)$ that has more than one global sink then there is a 2-dimensional face of P on which $\mathcal{O}$ induces more than one sink. (In other words, to check whether $\mathcal{O}$ is Kalai's "good orientation" on $G(P)$, it is enough to inspect 2-dimensional faces only! This result in fact holds for ALL simple polytopes, not just cubes, and in that generality is due to Joswig, Kaibel, and Körner.)
Hint: by induction on d one can assume that the two global sinks are antipodal vertices t_1 and t_2 of P. Consider a sink of the subgraph $G(A) \subset G(P)$ induced by $A := \text{vert}(P) - \{t_1, t_2\}$.

(3) A polytope P is called *k-neighborly* if every subset of k vertices of P is the set of vertices of a proper face of P.
  (a) Show that if P is k-neighborly then every k vertices of P are affinely independent.
  (b) Conclude that if P is k-neighborly and $1 \leq j \leq k$, then P is also j-neighborly and has exactly $\binom{|\text{vert}(P)|}{j}$ $(j-1)$-dimensional faces.
  (c) Show that if P is a k-neighborly n-dimensional polytope with $k > \lfloor n/2 \rfloor$, then P is a simplex. (**Hint:** use Radon's theorem)
  (d) Conclude that every n-neighborly $2n$-dimensional polytope is simplicial.