

- (1) Show that a  $t$ -collapsible complex is  $t$ -Leray.  
**(Hint:** How does a collapse affect the “higher” Betti numbers?)
- (2) Let  $P$  be a finite graded poset with ranks  $1, 2, \dots, d$ , and  $S$  a subset of  $[d]$ . Show that if  $P$  is shellable (i.e., its order complex  $\Delta(P)$  is shellable), then so is the rank-selected subposet  $P_S$ .  
 (Recall that a poset is called graded if all its maximal chains are of the same length.)
- (3) Let  $\Delta$  be a  $(d-1)$ -dimensional balanced Buchsbaum complex. (Recall that  $\Delta$  is Buchsbaum if it is pure and  $\tilde{H}_i(\text{lk } F) = 0$  for every non-empty face  $F$  of  $\Delta$  and all  $i < d - |F| - 1$ .)
- (a) Show that all rank-selected subcomplexes of  $\Delta$  are also Buchsbaum.
- (b) Show that  $\beta_i(\Delta_S) = \beta_i(\Delta)$  for all  $i < |S| - 1$ , and that  $\beta_{|S|-1}(\Delta_S) \geq \beta_{|S|-1}(\Delta)$ .  
**(Hint:** mimic one of the proofs we saw in class.)
- (c) Use part (b) and the fact that  $h_i(\Delta) = \sum_{|S|=i} h_i(\Delta_S)$  to conclude that

$$h_i(\Delta) \geq \binom{d}{i} \sum_{j=1}^i (-1)^{i-j} \beta_{j-1}(\Delta).$$

**Remark:** This inequality is known to hold for all (not only balanced) Buchsbaum simplicial complexes. However the proof in the general case is much more complicated.

- (4) Let  $\Delta$  be a Cohen-Macaulay complex, and  $v$  its vertex.
- (a) Show that  $h_i(\Delta) \geq h_i(\text{star } v)$  for all  $0 \leq i \leq d$ .  
 (Recall that  $\text{star } v = \{G \in \Delta : \{v\} \cup G \in \Delta\} = v * \text{lk } v$ .)
- (b) Conclude that  $h_i(\Delta) \geq h_i(\text{lk } v)$  for all  $0 \leq i \leq d - 1$ .  
**(Hint:** what is the relationship between  $h(K * L)$  and the  $h$ -vectors of  $K$  and  $L$ ?)