

- (1) For a positive real number $\alpha < 2$, let $B(n+1, \alpha)$ be the (infinite) Borsuk graph with S^n as the vertex set and with two points connected by an edge iff their distance is at least α . Prove that the Borsuk-Ulam theorem is equivalent to the following statement: for every $\alpha < 2$, we have $\chi(B(n+1, \alpha)) \geq n+2$ (here χ denotes the usual chromatic number.)
- (2) Prove that the mass distribution in the plane can be dissected into four equal parts by two lines.
- (3) Prove the planar case of the theorem on multicolored partitions by considering a perfect red-blue matching with the minimum possible total length of edges.
- (4) Call a subset $S \in \binom{[n]}{k}$ **stable** if it does not contain any two adjacent elements modulo n . (That is, if $i \in S$, then $i+1 \notin S$, and if $n \in S$, then $1 \notin S$. In other words, S corresponds to an independent set of a cycle C_n . We denote by $\binom{[n]}{k}_{\text{stab}}$ the family of stable k -subsets of $[n]$.)
- (a) Prove the following strengthening of Gale's lemma: *There exists a $(2k+d)$ -point set $X \subset S^d$ such that under a suitable identification of X with $[n]$, every open hemisphere contains a stable k -tuple.*
- Hint:** look carefully at the proof of Gale's lemma we saw in class.
- (b) The **Schrijver graph** is

$$SG_{n,k} := KG \left(\binom{[n]}{k}_{\text{stab}} \right).$$

It is an induced subgraph of the Kneser graph $KG_{n,k}$.

Prove that for all $n \geq 2k \geq 0$, we have

$$\chi(SG_{n,k}) = \chi(G_{n,k}) = n - 2k + 2.$$

(This result is due to Schrijver, 1978.)

Hint: use part (a).