

Face rings of complexes with singularities

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Abstract

It is shown that the face ring of a pure simplicial complex modulo m generic linear forms is a ring with finite local cohomology if and only if the link of every face of dimension m or more is nonsingular.

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1 Introduction

In the 70's Reisner, building on unpublished work of Hochster, and Stanley revolutionized the study of face enumeration of simplicial complexes through their use of the face ring, also called the Stanley-Reisner ring. Reisner proved that the face ring of a complex is Cohen-Macaulay if and only if the link of every face, including the empty face, is nonsingular [6]. Here, nonsingular means that all reduced cohomology groups, except possibly in the maximum dimension, vanish. Stanley used this to completely characterize f -vectors of such complexes [8].

A natural question which follows these results is, “What happens if singularities are allowed?” The weakest relaxation possible is to permit nontrivial cohomology in the lower dimensions of the whole complex (the link of the empty face). Schenzel proved that for pure

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complexes the face ring is Buchsbaum if and only if this is the case [7]. The primary tool used in the proof of both Reisner’s and Schenzel’s theorems is the local cohomology of the face ring with respect to the irrelevant ideal. In the Cohen–Macaulay case the local cohomology modules vanish below the top dimension, while in the Buchsbaum case these modules are finite-dimensional. Rings with this property, that is, those whose local cohomology modules below their Krull dimension are finite-dimensional, are called generalized Cohen–Macaulay rings or rings with finite local cohomology. The goal of this note is to extend these ideas to arbitrary singularities. Our main theorem says that if the dimension of the singular set is $m - 1$, then the the face ring modulo m generic linear forms is a ring with finite local cohomology. The precise statement and all definitions are in the next section. This is followed by a simple example in Section 3 and the proof of the main theorem in Section 4.

2 Preliminaries

For all undefined terminology we refer our readers to [1, 9]. Throughout Δ is a pure $(d - 1)$ -dimensional simplicial complex with vertex set $[n] = \{1, \dots, n\}$. If $F \in \Delta$ is a face, then the *link* of F is

$$\text{lk } F = \{G \in \Delta : F \cap G = \emptyset, F \cup G \in \Delta\}.$$

In particular, $\text{lk } \emptyset = \Delta$. We say that the face F is *nonsingular* if $\tilde{H}^i(\text{lk } F; \mathbf{k}) = 0$ for all $i < d - 1 - |F|$. Otherwise F is a *singular* face. The *singularity dimension* of Δ is the maximum dimension of a singular face. If there are no singular faces, then Δ is *Cohen–Macaulay* and we (arbitrarily) declare the singularity dimension of the complex to be $-\infty$.

For a field \mathbf{k} , which we will always assume is infinite (of arbitrary characteristic), the *face ring* of Δ (also known as the *Stanley–Reisner* ring) is

$$\mathbf{k}[\Delta] = \mathbf{k}[x_1, \dots, x_n]/I_\Delta,$$

where I_Δ is the ideal generated by

$$(x_{i_1} \cdots x_{i_k} : \{i_1, \dots, i_k\} \notin \Delta).$$

Let $\mathfrak{M} = (x_1, \dots, x_n)$ be the irrelevant ideal of $S = \mathbf{k}[x_1, \dots, x_n]$. For any S -module M we use $H_{\mathfrak{M}}^i(M)$ to denote the local cohomology modules of M with respect to \mathfrak{M} . If the Krull dimension of M is d , then we say that M is a *module with finite local cohomology* (or a *generalized Cohen–Macaulay* module) if for all $i < d$, $H_{\mathfrak{M}}^i(M)$ is finite-dimensional as a \mathbf{k} -vector space. Modules with finite local cohomology were originally introduced in [2], [10] and [11]. Connections between face rings and modules with finite local cohomology have been studied in [4] and [12]. The main goal of this note is to prove the following.

Theorem 2.1 *Let Δ be a pure $(d - 1)$ -dimensional complex. Then the singularity dimension of Δ is less than m if and only if for all sets of m generic linear forms $\{\theta_1, \dots, \theta_m\}$ the quotient $\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m)$ is a ring with finite local cohomology.*

The proof will rely on results of Gräbe [5] which we now explain. Denote by $|\Delta|$ the geometric realization of Δ . For a face $\tau \in \Delta$, let $\text{cost } \tau := \{\sigma \in \Delta : \sigma \not\supseteq \tau\}$ be the contrastar of τ , let $H^i(\Delta, \text{cost } \tau)$ be the simplicial i -th cohomology of the pair (with coefficients in \mathbf{k}), and for $\tau \subset \sigma \in \Delta$, let ι^* be the map $H^i(\Delta, \text{cost } \sigma) \rightarrow H^i(\Delta, \text{cost } \tau)$ induced by inclusion $\iota : \text{cost } \tau \rightarrow \text{cost } \sigma$. Finally, for a vector $U = (u_1, \dots, u_n) \in \mathbb{Z}^n$, let $s(U) := \{l : u_l \neq 0\} \subseteq [n]$ be the support of U , let $|U| = \sum_{l=1}^n u_l$, let $\{e_l\}_{l=1}^n$ be the standard basis for \mathbb{Z}^n , and let \mathbb{N} denote the set of nonnegative integers.

We consider the \mathbb{Z}^n -grading of $\mathbf{k}[x_1, \dots, x_n]$ obtained by declaring x_l to be of degree e_l . This grading refines the usual \mathbb{Z} -grading and induces a \mathbb{Z}^n -grading of $\mathbf{k}[\Delta]$ and its local cohomology modules. Thus, $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])_j = \bigoplus H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])_U$ where the sum is over all $U = (u_1, \dots, u_n) \in \mathbb{Z}^n$ with $|U| = j$, and multiplication by x_l is a linear map from $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])_U$ to $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])_{U+e_l}$ for all $U \in \mathbb{Z}^n$.

Theorem 2.2 [Gräbe] *The following is an isomorphism of \mathbb{Z}^n -graded $\mathbf{k}[\Delta]$ -modules*

$$H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]) \cong \bigoplus_{\substack{-U \in \mathbb{N}^n \\ s(U) \in \Delta}} \mathcal{M}_U^i, \quad \text{where } \mathcal{M}_U^i = H^{i-1}(\Delta, \text{cost } s(U)), \quad (1)$$

and the $\mathbf{k}[\Delta]$ -structure on the U -th component of the right-hand side is given by

$$\cdot x_l = \begin{cases} 0\text{-map}, & \text{if } l \notin s(U) \\ \text{identity map}, & \text{if } l \in s(U) \text{ and } l \in s(U + e_l) \\ \iota^* : H^{i-1}(\Delta, \text{cost } s(U)) \rightarrow H^{i-1}(\Delta, \text{cost } s(U + e_l)), & \text{otherwise.} \end{cases}$$

We note that the isomorphism of (1) on the level of vector spaces (rather than $\mathbf{k}[\Delta]$ -modules) is due to Hochster, see [9, Section II.4], and that

$$H^i(\mathbf{k}[\Delta])_0 \cong \mathcal{M}_{(0, \dots, 0)}^i = H^{i-1}(\Delta, \emptyset) = \tilde{H}^{i-1}(\Delta; \mathbf{k}).$$

3 Isolated singularities

Before proceeding to the proof of the main theorem we consider a special case. We say that Δ has *isolated singularities* if the singularity dimension of Δ is zero and there is at least one singular vertex. For the rest of the section we assume that Δ has isolated singularities.

To begin with, we compute $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])$ for $i < d$. Since Δ has isolated singularities Theorem 2.2 says that

$$H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])_U = \begin{cases} \tilde{H}^{i-1}(\Delta; \mathbf{k}), & s(U) = \emptyset \\ H^{i-1}(\Delta, \text{cost } \{j\}), & s(U) = \{j\}, U \in -\mathbb{N}^n \\ 0, & \text{otherwise.} \end{cases}$$

Let $\theta = \sum_{i=1}^n a_i x_i$ be a linear form in S with $a_i \neq 0$ for all i . In order to compute $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]/(\theta))$ for $i < d - 1$ (the Krull dimension of $\mathbf{k}[\Delta]/(\theta)$ is $d - 1$) we use the following short exact sequence

$$0 \rightarrow \mathbf{k}[\Delta] \xrightarrow{\theta} \mathbf{k}[\Delta] \rightarrow \mathbf{k}[\Delta]/(\theta) \rightarrow 0.$$

Since multiplication by $\theta, \cdot\theta$, is injective for any face ring, the above is in fact a short exact sequence. The corresponding long exact sequence in local cohomology is

$$\cdots \rightarrow H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]) \xrightarrow{(\cdot\theta)^*} H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]) \rightarrow H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]/(\theta)) \xrightarrow{\delta} H_{\mathfrak{M}}^{i+1}(\mathbf{k}[\Delta]) \xrightarrow{(\cdot\theta)^*} H_{\mathfrak{M}}^{i+1}(\mathbf{k}[\Delta]) \rightarrow \cdots$$

Here, δ is the connecting homomorphism which decreases the \mathbb{Z} -grading by one, and $\cdot\theta^*$ is the map induced by multiplication, and hence is just the module action of multiplication by θ on $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta])$. Let f^i be the map in simplicial cohomology

$$f^i : \left(\bigoplus_{t=1}^n H^i(\Delta, \text{cost } \{t\}) \right) \rightarrow H^i(\Delta, \emptyset), \quad f^i = \sum_{t=1}^n a_t \cdot \iota^* [H^i(\Delta, \text{cost } \{t\}) \rightarrow H^i(\Delta, \emptyset)].$$

By Gräbe's description of the S -module structure of the local cohomology modules (using the \mathbb{Z} -grading) and the above long exact sequence,

$$H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]/(\theta))_j \cong \begin{cases} 0, & j < 0 \\ \text{coker } f^{i-1} \oplus \ker f^i, & j = 0 \\ H^i(\Delta, \emptyset), & j = 1. \end{cases}$$

In particular, $H_{\mathfrak{M}}^i(\mathbf{k}[\Delta]/(\theta))$ is finite-dimensional.

4 Proof

We now proceed to the proof of the main theorem. As before, let Δ be a $(d-1)$ -dimensional simplicial complex on $[n]$. Let $\theta_1, \dots, \theta_d$ be d generic linear forms in S with $\theta_p = \sum_{t=1}^n a_{t,p} x_t$. In particular, we assume that every square submatrix of the $n \times d$ -matrix $A = (a_{t,p})$ is non-singular and the θ 's satisfy the prime avoidance argument in the proof of Lemma 4.3 below. Each θ_p acts on $\mathbf{k}[\Delta]$ by multiplication, $\cdot\theta_p : \mathbf{k}[\Delta] \rightarrow \mathbf{k}[\Delta]$. This map, in turn, induces the map $(\cdot\theta_p)^* = \cdot\theta_p : H_{\mathfrak{M}}^l(\mathbf{k}[\Delta]) \rightarrow H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])$ that increases the \mathbb{Z} -grading by one. The key objects in the proof are the kernels of these maps and their intersections:

$$\ker_{m,i}^l := \bigcap_{p=1}^m (\ker(\cdot\theta_p)^* : H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])_{-(i+1)} \rightarrow H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])_{-i}) \quad \text{and} \quad \ker_m^l := \bigoplus_{i \in \mathbb{Z}} \ker_{m,i}^l.$$

Thus \ker_m^l is a graded submodule of $H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])$ and $\ker_{m,i}^l$ is simply $(\ker_m^l)_{-(i+1)}$.

What are the dimensions of these kernels? If $m = 0$, then $\ker_{0,i}^l = H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])_{-(i+1)}$, and Theorem 2.2 implies that for $i \geq 0$,

$$\begin{aligned} \dim_{\mathbf{k}} \ker_{0,i}^l &= \sum_{F \in \Delta} |\{U \in \mathbb{N}^n : s(U) = F, |U| = i+1\}| \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F) \\ &= \sum_{F \in \Delta} \binom{i}{|F|-1} \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F). \end{aligned} \tag{2}$$

For a general m , we prove the following (where we set $\binom{a}{b} = 0$ if $b < 0$).

Lemma 4.1 For every $0 \leq m \leq d$, $l \leq d$, and $i \geq m$,

$$\dim_{\mathbf{k}} \ker_{m,i}^l \leq \sum_{F \in \Delta} \binom{i-m}{|F|-m-1} \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F).$$

In fact, using Lemma 4.1 (but deferring its proof to the end of the section), we can say even more:

Lemma 4.2 For every $0 \leq m \leq d$, $l \leq d$, and $i \geq m$,

$$\dim_{\mathbf{k}} \ker_{m,i}^l = \sum_{F \in \Delta} \binom{i-m}{|F|-m-1} \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F).$$

Moreover, the map $\oplus_{i \geq m+1} \ker_{m,i}^l \xrightarrow{\cdot \theta_{m+1}} \oplus_{i \geq m+1} \ker_{m,i-1}^l$, is a surjection.

Proof: We prove the statement on the dimension of $\ker_{m,i}^l$ by induction on m . For $m = 0$ (and any $l, i \geq 0$), this is eq. (2). For larger m , we notice that the restriction of $(\cdot \theta_{m+1})^*$ to $\ker_{m,i}^l$ is a linear map from $\ker_{m,i}^l$ to $\ker_{m,i-1}^l$, whose kernel is $\ker_{m+1,i}^l$. Thus for $i \geq m+1$,

$$\begin{aligned} \dim_{\mathbf{k}} \ker_{m+1,i}^l &\geq \dim_{\mathbf{k}} \ker_{m,i}^l - \dim_{\mathbf{k}} \ker_{m,i-1}^l \\ &= \sum_{F \in \Delta} \left[\binom{i-m}{|F|-m-1} - \binom{i-1-m}{|F|-m-1} \right] \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F) \\ &= \sum_{F \in \Delta} \binom{i-(m+1)}{|F|-(m+1)-1} \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F). \end{aligned}$$

The second step in the above computation is by the inductive hypothesis. Comparing the resulting inequality to that of Lemma 4.1 shows that this inequality is in fact equality, and hence that the map $(\cdot \theta_{m+1})^* : \ker_{m,i}^l \rightarrow \ker_{m,i-1}^l$ is surjective for $i \geq m+1$. \square

Lemma 4.2 allows us to get a handle on $H_{\mathfrak{M}}^l(\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m))_{-i}$ at least for $l, i > 0$:

Lemma 4.3 For $0 \leq m \leq d$ and $0 < l \leq d - m$, there is a graded isomorphism of modules

$$\bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m))_{-i} \cong \bigoplus_{i \geq 1} \ker_{m,i+m-1}^{l+m} \quad \text{with} \quad H_{\mathfrak{M}}^l(\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m))_{-i} \cong \ker_{m,i+m-1}^{l+m}.$$

Thus, for $l, i > 0$,

$$\dim_{\mathbf{k}} H_{\mathfrak{M}}^l(\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m))_{-i} = \sum_{F \in \Delta} \binom{i-1}{|F|-m-1} \cdot \dim_{\mathbf{k}} H^{l+m-1}(\Delta, \text{cost } F).$$

Proof: The proof is by induction on m , with the $m = 0$ case being evident. For larger m , we want to mimic the proof given in Section 3. One obstacle to this approach is that the map $\cdot\theta_{m+1} : \mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m) \rightarrow \mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m) =: \mathcal{M}[m]$ might not be injective anymore. However, a “prime avoidance” argument together with the genericity assumption on θ_{m+1} implies that $\cdot\theta_{m+1} : \mathcal{M}[m]/H_{\mathfrak{M}}^0(\mathcal{M}[m]) \rightarrow \mathcal{M}[m]/H_{\mathfrak{M}}^0(\mathcal{M}[m])$ is injective, and hence one has the corresponding long exact sequence in local cohomology. See, for instance, [3, Chapter 3] for details on “prime avoidance” arguments. Since $H_{\mathfrak{M}}^0(\mathcal{M}[m])$ has Krull dimension 0, modding $H_{\mathfrak{M}}^0(\mathcal{M}[m])$ out does not affect $H_{\mathfrak{M}}^l$ for $l > 0$, so that the part of this long exact sequence corresponding to $l, i > 0$ can be rewritten as

$$\begin{array}{ccc} \bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathcal{M}[m])_{-(i+1)} & \xrightarrow{(\cdot\theta_{m+1})^*} & \bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathcal{M}[m])_{-i} \longrightarrow \bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathcal{M}[m+1])_{-i} \\ & \searrow \delta & \bigoplus_{i \geq 1} H_{\mathfrak{M}}^{l+1}(\mathcal{M}[m])_{-(i+1)} \xrightarrow{(\cdot\theta_{m+1})^*} \bigoplus_{i \geq 1} H_{\mathfrak{M}}^{l+1}(\mathcal{M}[m])_{-i}. \end{array}$$

By the inductive hypothesis combined with Lemma 4.2, the leftmost map in this sequence is surjective. Hence the module $\bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathcal{M}[m+1])_{-i}$ is isomorphic to the kernel of the rightmost map. Applying the inductive hypothesis to the last two entries of the sequence then implies the following isomorphism of modules:

$$\bigoplus_{i \geq 1} H_{\mathfrak{M}}^l(\mathcal{M}[m+1])_{-i} \cong \bigoplus_{i \geq 1} \ker((\cdot\theta_{m+1})^* : \ker_{m, i+m}^{l+m+1} \rightarrow \ker_{m, i+m-1}^{l+m+1}) = \bigoplus_{i \geq 1} \ker_{m+1, i+m}^{l+m+1}.$$

□

Theorem 2.1 now follows easily from Lemma 4.3:

Proof of Theorem 2.1: Since $\mathbf{k}[\Delta]$ is a finitely-generated algebra, $H_{\mathfrak{M}}^0(\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m))$ has Krull dimension zero, and hence is a finite-dimensional vector space for any simplicial complex Δ . So we only need to care about $H_{\mathfrak{M}}^l$ for $l > 0$. As $\binom{i-1}{|F|-m-1} > 0$ for all $i \gg 0$ and $|F| > m$, Lemma 4.3 implies that $\mathbf{k}[\Delta]/(\theta_1, \dots, \theta_m)$ is a ring with finite local cohomology if and only if for all faces $F \in \Delta$ of size larger than m and all $l + m < d$, the cohomology $H^{l+m-1}(\Delta, \text{cost } F)$ vanishes. Given that $H^{l+m-1}(\Delta, \text{cost } F)$ is isomorphic to $\tilde{H}^{l+m-1-|F|}(\text{lk } F; \mathbf{k})$ (see e.g. [5, Lemma 1.3]), this happens if and only if each such F is non-singular. □

To finish the proof of Theorem 2.1, it only remains to verify Lemma 4.1. For its proof we use the following notation. Fix $m, l > 0$, and $i \geq m$. For $r \in \{i, i+1\}$, let $\mathbf{V}_r := \{U \in \mathbb{N}^n : |U| = r, s(U) \in \Delta\}$, and for $F \in \Delta$, let $\mathbf{V}_{r,F} := \{u \in \mathbf{V}_r : s(U) = F\}$. If $F = \{f_1 < \dots < f_j\} \in \Delta$ where $j > m$, then set $\mathbf{W}_{r,F} := \{U = (u_1, \dots, u_n) \in \mathbf{V}_{r,F} : u_{f_s} = 1 \text{ for } 1 \leq s \leq m\}$. Observe that $\mathbf{W}_{i+1,F}$ is a subset of $\mathbf{V}_{i+1,F}$ of cardinality $\binom{i-m}{|F|-m-1}$.

For $G \subseteq F \in \Delta$ let $\Phi_{F,G}$ denote the map $\iota^* : H^{l-1}(\Delta, \text{cost } F) \rightarrow H^{l-1}(\Delta, \text{cost } G)$. Thus, $\Phi_{F,G}$ is the identity map if $F = G$. Using Theorem 2.2, we identify $H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])_{-r}$ with $\bigoplus_{U \in \mathbf{V}_r} H^{l-1}(\Delta, \text{cost } s(U))$, and for $z \in H_{\mathfrak{M}}^l(\mathbf{k}[\Delta])_{-r}$ we write $z = (z_U)_{U \in \mathbf{V}_r}$, where

$z_U \in H^{l-1}(\Delta, \text{cost } s(U))$ is the $(-U)$ -th component of z . Since $\theta_p = \sum_{t=1}^n a_{t,p} x_t$, Theorem 2.2 yields that for such z , $r = i + 1$, and $T \in \mathbf{V}_i$,

$$(\theta_p z)_T = \sum_{\{t: T+e_t \in \mathbf{V}_{i+1}\}} a_{t,p} \cdot \Phi_{s(T+e_t), s(T)}(z_{T+e_t}). \quad (3)$$

Proof of Lemma 4.1: Since $|\mathbf{W}_{i+1, F}| = \binom{i-m}{|F|-m-1}$ for all $F \in \Delta$, to prove that $\dim_{\mathbf{k}} \ker^l_{m,i} \leq \sum_{F \in \Delta} \binom{i-m}{|F|-m-1} \cdot \dim_{\mathbf{k}} H^{l-1}(\Delta, \text{cost } F)$, it is enough to verify that for $z, z' \in \ker^l_{m,i}$,

$$z_U = z'_U \text{ for all } F \in \Delta \text{ and all } U \in \mathbf{W}_{i+1, F} \implies z = z',$$

or equivalently (since $\ker^l_{m,i}$ is a \mathbf{k} -space) that for $z \in \ker^l_{m,i}$,

$$z_U = 0 \text{ for all } F \in \Delta \text{ and all } U \in \mathbf{W}_{i+1, F} \implies z = 0. \quad (4)$$

To prove this, fix such a z . From eq. (3) and the definition of $\ker^l_{m,i}$, it follows that

$$\sum_{\{t: T+e_t \in \mathbf{V}_{i+1}\}} a_{t,p} \cdot \Phi_{s(T+e_t), s(T)}(z_{T+e_t}) = 0 \quad \forall 1 \leq p \leq m \text{ and } \forall T \in \mathbf{V}_i. \quad (5)$$

For a given $T \in \mathbf{V}_i$, we refer to the m conditions imposed on z by eq. (5) as “the system defined by T ”, and denote this system by \mathcal{S}_T .

Define a partial order, \succ , on \mathbf{V}_i as follows: $T' \succ T$ if either $|s(T')| > |s(T)|$, or $s(T') = s(T)$ and the last non-zero entry of $T' - T$ is positive. To finish the proof, we verify by a descending (w.r.t \succ) induction on $T \in \mathbf{V}_i$, that $z_{T+e_t} = 0$ for all t with $T + e_t \in \mathbf{V}_{i+1}$. For $T \in \mathbf{V}_i$, there are two possible cases (we assume that $s(T) = F = \{f_1 < \dots < f_j\}$):

Case 1: $T \in \mathbf{W}_{i, F}$. Then for each $t \notin F$, either $F' := F \cup \{t\} \notin \Delta$, in which case $T + e_t \notin \mathbf{V}_{i+1}$, or $T + e_t \in \mathbf{W}_{i+1, F'}$, in which case $z_{T+e_t} = 0$ by eq. (4). Similarly, if $t = f_r$ for some $r > m$, then $T + e_t \in \mathbf{W}_{i+1, F}$, and $z_{T+e_t} = 0$ by (4). Finally, for any $t \in F$, $s(T + e_t) = s(T) = F$, and so $\Phi_{s(T+e_t), s(T)}$ is the identity map. Thus the system \mathcal{S}_T reduces to m linear equations in m variables:

$$\sum_{r=1}^m a_{f_r, p} \cdot z_{T+e_{f_r}} = 0 \quad \forall 1 \leq p \leq m. \quad (6)$$

Since the matrix $(a_{f_r, p})_{1 \leq r, p \leq m}$ is non-singular, $(z_{T+e_t} = 0 \text{ for all } t)$ is the only solution of \mathcal{S}_T .

Case 2: $T \notin \mathbf{W}_{i, F}$, and so $T_{f_s} \geq 2$ for some $s \leq m$. Then for any $t \notin F$, $T' := T - e_{f_s} + e_t \succ T$, as T' has a larger support than T , and $T' + e_{f_s} = T + e_t$. Hence $z_{T+e_t} = z_{T'+e_{f_s}} = 0$ by the inductive hypothesis on T' . Similarly, if $t \in F - \{f_1, \dots, f_m\}$, then $t > f_s$, and so $T'' := T - e_{f_s} + e_t \succ T$. As $T'' + e_{f_s} = T + e_t$, the inductive hypothesis on T'' imply that $z_{T+e_t} = 0$. Thus, as in Case 1, \mathcal{S}_T reduces to system (6), whose only solution is trivial. \square

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