Math 124 Section H, Autumn 2014
Midterm Exam Number Two: Solutions

1. (a) Straightforward chain rule stuff. Remember that the derivative of 4^x is $\ln(4)4^x$, so:

$$y' = \ln(4)4^{2x+5}(2)$$

(b) Use the chain rule a couple times. Be careful about the parentheses!

$$\frac{dy}{dx} = -\sin\left(x - \arctan(\sqrt{x+1})\right)\left(1 - \frac{1}{1+(\sqrt{x+1})^2}\left(\frac{1}{2\sqrt{x+1}}\right)\right)$$

(c) Take the natural log of both sides, then rewrite as $\ln(y) = x^x \ln(x)$. Two ways to go from here: either separately figure out the derivative of x^x, or take the natural log again: $\ln(\ln(y)) = \ln(x^x \ln(x)) = x\ln(x) + \ln(\ln(x))$
Then we can differentiate both sides to get:

$$\frac{\frac{y'}{y}}{\ln(y)} = \ln(x) + 1 + \frac{\frac{1}{x}}{\ln(x)}$$

Or:

$$\frac{y'}{x^{x^x}\ln\left(x^{x^x}\right)} = \ln(x) + 1 + \frac{1}{x\ln(x)}$$

So:

$$y' = \left(x^{x^x}\ln\left(x^{x^x}\right)\right)\left(\ln(x) + 1 + \frac{1}{x\ln(x)}\right)$$

2. Suppose $(x(t), y(t)) = (e^{3t}+4, \sin(3t))$ is the point of tangency. Then we can calculate the slope of the tangent line in two ways: once by taking the slope between the above and $(4,0)$, and again by finding $y'(t)/x'(t)$. Set those equal and you've got:

$$\frac{\sin(3t) - 0}{e^{3t}+4-4} = \frac{3\cos(3t)}{3e^{3t}}$$

Which means:

$$\frac{\sin(3t)}{\cos(3t)} = 1$$

So $\tan(3t) = 1$, so $3t = \pi/4$ and $t = \pi/12$. Plug that in and you get a slope of $\frac{\sqrt{2}}{2e^{\pi/4}}$ so the tangent line is:

$$y = \frac{\sqrt{2}}{2e^{\pi/4}}(x-4) + 0$$

3. Okay, we want to use a tangent line approximation at $\beta = 0.600256$. That seems pretty close to $\beta = 0.6$, and good news: the value at $\beta = 0.6$ is pretty easy to calculate:

$$f_o = 400\sqrt{\frac{1-.6}{1+.6}} = 400\sqrt{\frac{0.4}{1.6}} = 400\sqrt{\frac{1}{4}} = 200 \text{ THz}$$

So all we need for the tangent line approximation is the slope at $\beta = 0.6$. Let's take the derivative (with respect to β), and we've got (using the chain rule and the quotient rule):

$$\frac{df_o}{d\beta} = 400\frac{1}{2\sqrt{\frac{1-\beta}{1+\beta}}}\frac{-(1+\beta)-(1-\beta)}{(1+\beta)^2}$$

Plug in $\beta = 0.6$ and you've got:

$$\frac{df_o}{d\beta} = 400\frac{1}{2\sqrt{\frac{1}{4}}}\frac{-2}{1.6^2} = \frac{-800}{1.6^2} = \frac{-800}{2.56}$$

So the tangent line approximation is $f_o \approx 200 + \dfrac{-800}{2.56}(\beta - 0.6)$, and at $\beta = 0.600256$ we get $f_o \approx 200 - 800(0.000256)/2.56 = 199.92$.

4. Oh, good, it's a closed interval and the function is continuous on that interval. So we should find some critical numbers and plug them in to find the minimum and maximum.

   We'll need $f'(x)$:

$$f'(x) = \frac{2(x-2)(x+2)+(x-2)^2}{2\sqrt{(x-2)^2(x+2)}} = \frac{3x^2-4x-4}{2\sqrt{(x-2)^2(x+2)}} = \frac{(3x+2)(x-2)}{2\sqrt{(x-2)^2(x+2)}}$$

   So the critical numbers are $x = -2$ and $x = 2$ (where $f'(x)$ is undefined), as well as $x = -2/3$.

   Plugging everything (including the endpoints) into $f(x) = \sqrt{(x-2)^2(x+2)}$ we get:
   $f(-2) = 0$

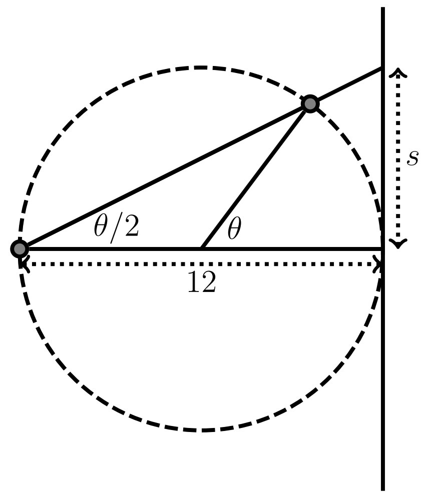
   $f(2) = 0$

   $f(3) = \sqrt{5}$

   $f(-2/3) = \sqrt{(-8/3)^2(4/3)} = \sqrt{256/27} = \sqrt{9.\text{something}}$ which is more than $\sqrt{5}$.

   So $\sqrt{256/27}$ is the maximum.

5. See a modified picture on the next page:



We know that Nick makes a complete lap every 25 seconds, so $\frac{d\theta}{dt} = \frac{-2\pi}{25}$ radians per second. So:

$$\tan\left(\frac{\theta}{2}\right) = \frac{s}{12}$$

Differentiate to get

$$\sec^2\left(\frac{\theta}{2}\right)\frac{1}{2}\frac{d\theta}{dt} = \frac{1}{12}\frac{ds}{dt}$$

When Nick is 3 feet from the wall, $\cos(\theta) = 1/2$, so $\theta = \pi/3$ and $\sec^2\left(\frac{\theta}{2}\right) = \frac{4}{3}$. So finally:

$$\frac{ds}{dt} = 12\sec^2\left(\frac{\theta}{2}\right)\frac{1}{2}\frac{-2\pi}{25} = \frac{-16\pi}{25}$$

So Nick's shadow moves at $16\pi/25$ feet per second.