Name: Solutions

Mathematics 307I Exam

Solutions

1. Solve these differential equations and initial value problems. Be sure to check your work.

   (a) $y' = \dfrac{1+t}{y}$

   Solution. This is separable, so separate the variables and integrate: $y\,dy = (1+t)dt$, so

$$\frac{1}{2}y^2 = t + \frac{1}{2}t^2 + c$$
$$y^2 = 2t + t^2 + c$$
$$\boxed{y = \pm\sqrt{2t + t^2 + c}}$$

   (b) $y' - 2y = e^t$, $y(0) = 1$

   Solution. This is first order linear, and the coefficient of y is -2, so the integrating factor is $e^{\int(-2)dt} = e^{-2t}$. Multiply by that and integrate:

$$e^{-2t}y' - 2e^{-2t}y = e^{-t}$$
$$e^{-2t}y = \int e^{-t}dt$$
$$e^{-2t}y = -e^{-t} + c$$
$$y = -e^t + ce^{2t}$$

   Now apply the initial condition: plug in $t = 0$ and $y = 1$: $1 = -1 + c$, so $c = 2$ and the answer is $\boxed{y = -e^{-t} + 2e^{2t}}$.

   (c) $y' + \dfrac{1}{t}y = ty^2$

   Solution. This is a Bernoulli equation and the right hand side has y^2, so first make the substitution $u = y^{1-2} = y^{-1}$, so $y = u^{-1}$. Then $u' = -y^{-2}y'$, or equivalently, $y' = -y^2u' = -u^{-2}u'$. After these substitutions, the equation becomes

$$-u^{-2}u' + \frac{1}{t}u^{-1} = tu^{-2}.$$

   This is supposed to be a linear equation, so try to put it into standard form: multiply both sides by $-u^2$ to make the coefficient of u' equal to 1:

$$u' - \frac{1}{t}u = -t.$$

   This is indeed linear, and the integrating factor is $e^{\int(-1/t)dt} = e^{-\ln t} = 1/t$. So multiply by that and integrate:

$$\frac{1}{t}u' - \frac{1}{t^2}u = -1$$
$$\frac{1}{t}u = -\int(1)dt = -t + c$$
$$u = -t^2 + ct.$$

Finally, $y = 1/u$, so

$$\boxed{y = \frac{1}{-t^2 + ct}}.$$

2. (a) Suppose you have a tank containing 100 liters of pure water. At time $t = 0$, you start two pumps; one adds salt water with concentration 5 grams/liter into the tank, and the other removes the resulting mixture from the tank. Each pumps at a rate of 2 liters/minute. Let $x(t)$ be the amount, in grams, of salt in this tank at time t. Find a formula for $x(t)$.

Solution. Since stuff is being pumped in at the same rate it's being pumped out, the volume stays constant at 100 liters. The concentration is the mass divided by volume, and so this is $x/100$. The rate of change for x is given by

$$\begin{aligned}\frac{dx}{dt} &= \text{(rate of salt flowing in)} - \text{(rate of salt flowing out)}\\ &= \text{(rate in liters per minute)} \cdot \text{(concentration of inflow)}\\ &\qquad - \text{(rate in liters per minute)} \cdot \text{(concentration of outflow)}\\ &= 2 \cdot 5 - 2 \cdot x/100\\ &= 10 - x/50.\end{aligned}$$

So there's a differential equation: $x' = 10 - x/50$. This is separable, and it's also linear, so I have a choice about how to solve it. I'll do it both ways, just for kicks.

Separable: $\frac{dx}{10-x/50} = dt$. Integrate both sides:

$$\begin{aligned}-50\ln(10 - x/50) &= t + c\\ \ln(10 - x/50) &= -t/50 + c\\ 10 - x/50 &= e^{-t/50+c} = Ae^{-t/50}\\ x/50 &= 10 - Ae^{-t/50}\\ x &= 500 + Be^{-t/50}.\end{aligned}$$

Now use the initial condition – when $t = 0$, $x = 0$ – to find that $B = -500$. So the answer is $\boxed{x = 500 - 500e^{-t/50}}$.

Linear: $x' + x/50 = 10$. The integrating factor is $e^{t/50}$, so multiply by that:

$$e^{t/50}x' + 2e^{t/50}x = 10e^{t/50}.$$

Now integrate and solve for x:

$$\begin{aligned}e^{t/50}x &= 500e^{t/50} + c\\ x &= 500 + ce^{t/50}\end{aligned}$$

Now use the initial condition – when $t = 0$, $x = 0$ – to find that $c = -500$. So the answer is $\boxed{x = 500 - 500e^{-t/50}}$.

(b) Call the tank in part (a) "tank 1." Suppose you also have a second tank, "tank 2," containing 200 liters of pure water at time $t = 0$, and the *output* from tank 1 is the *input* for tank

2. You also pump the mixture out of the second tank at 2 liters/minute. Let $y(t)$ be the amount (in grams) of salt in tank 2 at time t. Find a formula for $y(t)$. Compute the limit, as $t \to \infty$, of the *concentration* of salt in tank 2. Does your answer make sense?

Solution. As in part (a), the volume stays constant, at 200 liters. So whatever y is, the concentration of salt in the water is $y/200$. The water is flowing in at a rate of 2 liters per minute, with concentration equal to the concentration of tank 1: $x/100$. The water is flowing out at 2 liters per minute, with the concentration of tank 2: $y/200$. So the rate of change of y is

$$\begin{aligned}\frac{dy}{dt} &= (\text{rate in}) - (\text{rate out}) \\ &= 2x/100 - 2y/200 \\ &= x/50 - y/100.\end{aligned}$$

That's the differential equation: $y' = x/50 - y/100$. Furthermore, we can (and should) use the formula for x from part (a): $x = 500 - 500e^{-t/50}$. So the differential equation is really

$$y' = \frac{500 - 500e^{-t/50}}{50} - y/100 = 10 - 10e^{-t/50} - \frac{1}{100}y.$$

This is a linear equation; in standard form, it looks like:

$$y' + \frac{1}{100}y = 10 - 10e^{-t/50}.$$

The integrating factor is $e^{t/100}$, so multiply by that and integrate:

$$\begin{aligned}e^{t/100}y &= 10\int e^{t/100}dt - 10\int e^{-t/100}dt \\ e^{t/100}y &= 1000e^{t/100} + 1000e^{-t/100} + c \\ y &= 1000 + 1000e^{-t/50} + ce^{-t/100}\end{aligned}$$

The initial condition – $y(0) = 0$ – says that $c = -2000$, so the equation is

$$\boxed{y = 1000 + 1000e^{-t/50} - 2000e^{-t/100}}.$$

To get the concentration, divide by the volume (200 liters):

$$\text{concentration} = 5 + 5e^{-t/50} - 10e^{-t/100}.$$

As t goes to infinity, this approaches $\boxed{5}$ because both of the exponential terms go to zero. This makes sense: we're pumping salt into tank 1 at a concentration of 5 grams per liter, so the concentration in tank 1 will eventually get close to 5. Then since we're pumping the mixture out of tank 1 into tank 2, the concentration in tank 2 should also approach 5.

3. (a) Describe Euler's method for approximating solutions to differential equations.

Solution. Suppose I'm looking at the differential equation $y' = f(t,y)$, and I have a step size of h and initial values of t_0 and y_0. At each stage, I increment t by h, and I increment y by $hf(t,y)$, since the slope is given by $y' = f(t,y)$. In equations:

$$\begin{aligned}t_k &= t_{k-1} + h, \\ y_k &= y_{k-1} + hf(t_{k-1}, y_{k-1}).\end{aligned}$$

Then y_k is the approximate value of the solution y when $t = t_k$.

(b) Consider the initial value problem $y' = \sin(\pi t + \frac{\pi y}{2})$, $y(0) = 1$. Use Euler's method to approximate $y(1)$ and $y(2)$. Use the step size $h = 1$.

Solution. We start with $t = 0$ and $y = 1$. First step:

$$t = 0 + 1 \quad \text{(increment by } h\text{)}$$
$$y = 1 + 1 \cdot \sin(\pi \cdot 0 + \frac{\pi \cdot 1}{2}) = 1 + \sin(\pi/2) = 2.$$

So $\boxed{y(1) \approx 2}$.

Second step:

$$t = 1 + 1$$
$$y = 2 + \sin(\pi \cdot 1 + \frac{\pi \cdot 2}{2}) = 2 + \sin(\pi + \pi) = 2.$$

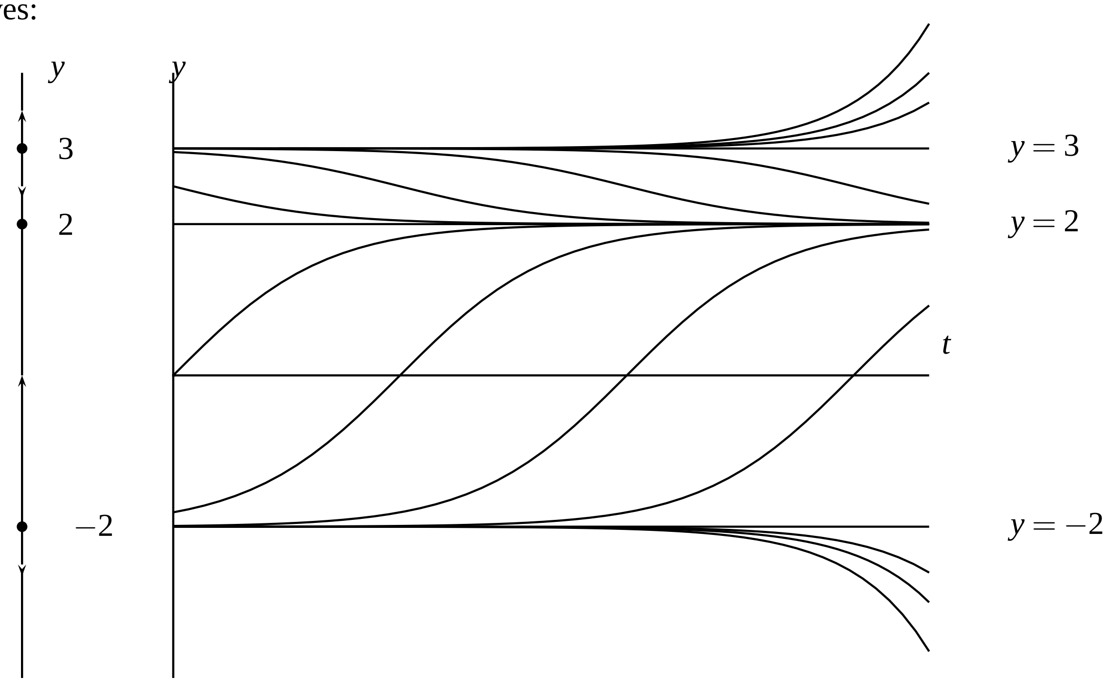
So $\boxed{y(2) \approx 2}$.

4. State Euler's formula (the one about complex numbers).

Solution. $\boxed{e^{it} = \cos t + i \sin t}$.

5. Consider the differential equation $y' = (3 - y)(4 - y^2)$. Determine the equilibrium points, classify each one as asymptotically stable, semistable, or unstable, and sketch some of the solution curves.

Solution. The right side is zero when $y = 3$ and when $4 - y^2 = 0$ – that is, when $y = \pm 2$. So these are the equilibrium points: $\boxed{y = -2,\ y = 2,\ y = 3}$. Here's a diagram of the y-axis, showing where y is increasing and decreasing, as well as some rough sketches of solution curves:



From these graphs, I can see that $y = -2$ is unstable, $y = 2$ is asymptotically stable, and $y = 3$ is unstable.

6. (5 bonus points) A projectile is subject to the forces of gravity and air resistance. In this problem, I'll assume that air resistance is proportional to the square of the velocity (up to a sign), and the relevant differential equation is

$$mv' = -mg - kv|v|$$

where m is the mass, g is gravity, and k is some positive constant. The limit $\lim_{t\to\infty} v$ is the *terminal velocity* of the object. This is an autonomous equation; use qualitative analysis to find the terminal velocity.

Solution. Divide both sides by m to get

$$v' = -g - \frac{k}{m}v|v|.$$

The right-hand side is zero when $-mg = kv|v|$, which is when $v|v| = -mg/k$. When v is positive, $|v|$ equals v and this says $v^2 = -mg/k$. This can't happen: v^2 is positive and $-mg/k$ is negative. When v is negative, $|v| = -v$, so this says $-v^2 = -mg/k$. I can solve this: $v^2 = \sqrt{mg/k}$. Since v is negative, this means

$$v = -\sqrt{\frac{mg}{k}}$$

is the only equilibrium point. When v is less than this, v' is positive; when v is bigger than this, v' is negative. Thus this is asymptotically stable, so the solutions approach it:

$$\boxed{\lim_{t\to\infty} v = -\sqrt{\frac{mg}{k}}}.$$