# Mathematics 307I Exam 2
## Solutions

1.

(a) Find the general solution of $y'' - 6y' + 9y = 0$.

**Solution.** The characteristic equation is $r^2 - 6r + 9 = 0$, which is the same as $(r-3)^2 = 0$. So this has one real root, $r = 3$. Therefore the general solution is $\boxed{y = c_1 e^{3t} + c_2 t e^{3t}}$.

(b) Find the solution of $y'' + 4y' = 0$, $y(0) = 1$, $y'(0) = 1$.

**Solution.** The characteristic equation is $r^2 + 4r = 0$ (*not* $r^2 + 4 = 0$), which can be written $r(r+4) = 0$. So this has two roots, $r = 0$ and $r = -4$. Since $e^{0t} = 1$, I can write the general solution as $y = c_1 + c_2 e^{-4t}$. I need to compute the derivative in order to apply the initial conditions: $y' = -4c_2 e^{-4t}$. The initial conditions say $1 = c_1 + c_2$ and $1 = 4c_2$. So $c_2 = -1/4$ and $c_1 = 5/4$, and the answer is $\boxed{y = \frac{5}{4} - \frac{1}{4} e^{-4t}}$.

(c) Find the general solution of $y'' + 3y' + 4y = -2e^{2t}$.

**Solution.** First I'll find $y_h$. The characteristic equation is $r^2 + 3r + 4 = 0$, which has roots $r = \dfrac{-3 \pm \sqrt{9-16}}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{7}}{2} i$. So $y_h = c_1 e^{-3t/2} \cos(\sqrt{7}t/2) + c_2 e^{-3t/2} \sin(\sqrt{7}t/2)$.

Now I'll find $y_p$. I'll try $y_p = Ae^{2t}$ – this is not part of $y_h$, so it will work. Then $y_p' = 2Ae^{2t}$ and $y_p'' = 4Ae^{2t}$. Plug this in: $4Ae^{2t} + 6Ae^{2t} + 4Ae^{2t} = -2e^{2t}$. So $14A = -2$, so $A = -1/7$, and $y_p = -\dfrac{1}{7} e^{2t}$.

So the general solution is

$$\boxed{y = c_1 e^{-3t/2} \cos\left(\frac{\sqrt{7}}{2} t\right) + c_2 e^{-3t/2} \sin\left(\frac{\sqrt{7}}{2} t\right) - \frac{1}{7} e^{2t}}.$$

2. Here is a nonhomogeneous differential equation:

$$y'' - y' - 2y = g(t).$$

(a) What is $y_h$, the solution of the associated homogeneous equation?

**Solution.** The characteristic equation is $r^2 - r - 2 = 0$, which factors as $(r-2)(r+1) = 0$. So the roots are $r = 2$ and $r = -1$, and $\boxed{y_h = c_1 e^{2t} + c_2 e^{-t}}$.

(b) If $g(t) = 7 \sin 4t$, what should you try for $y_p$?

**Solution.** Since $\sin 4t$ is not part of $y_h$, I'll try $\boxed{y_p = A \sin 4t + B \cos 4t}$.

(c) If $g(t) = 6e^{-t}$, what should you try for $y_p$?

**Solution.** Since $y_h$ contains $e^{-t}$, $y_p = Ae^{-t}$ will not work. So I should use $\boxed{y_p = Ate^{-t}}$.

(d) If $g(t) = -3e^{-2t} + t^3$, what should you try for $y_p$?

**Solution.** Neither $e^{-2t}$ nor a polynomial are part of $y_h$, so I'll try something times $e^{-2t}$ plus a degree three polynomial: $\boxed{y_p = Ae^{-2t} + Bt^3 + Ct^2 + Dt + E}$.

(e) If $g(t) = 2t^2 e^t \cos 3t$, what should you try for $y_p$?

**Solution.** $g(t)$ is a degree two polynomial times exponential times trig, so I would use

$$\boxed{y_p = (At^2 + Bt + C)e^t \cos 3t + (Dt^2 + Et + F)e^t \sin 3t}.$$

Since $y_h$ is nowhere close to this, this will work.

3.

(a) Here is the equation of an undamped mass-spring system with an external force acting on it: $2u'' + 8u = 5\cos \omega t$. For what value(s) of $\omega$ will the system exhibit resonance?

**Solution.** You get resonance when the driving frequency $\omega$ is equal to the natural frequency $\omega_0$. To find $\omega_0$, you can either remember that it's equal to $\sqrt{k/m}$, which is $\sqrt{8/2} = \sqrt{4} = 2$ in this case, or you can solve the associated homogeneous equation and find that $y_h = c_1 \sin 2t + c_2 \cos 2t$ – $y_h$ oscillates with frequency 2, so $\omega_0 = 2$. In any case, you get resonance when $\boxed{\omega = \omega_0 = 2}$.

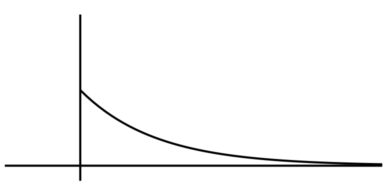
(b) Here is the equation of a damped mass-spring system with no external force acting on it: $3u'' + \gamma u' + u = 0$. For what value(s) of $\gamma$ will the mass exhibit oscillations?

**Solution.** You get oscillations when the solution has sines or cosines in it, not just exponentials. Therefore you get oscillations when the roots of the characteristic equation are complex, not real. This happens when the term under the square root in the quadratic formula is negative: $\gamma^2 - 12 < 0$. This is equivalent to $\gamma^2 < 12$. This means that $\boxed{-\sqrt{12} < \gamma < \sqrt{12}}$. In a mass-spring system, $\gamma$ is never negative, so you could also say $\boxed{0 \le \gamma < \sqrt{12}}$.

(c) Here is the equation of a damped mass-spring system with no external force acting on it: $mu'' + 4u' + u = 0$. Suppose $m$ gradually changes from 1 to 10. How do the solutions of the equation change? Be specific in your answer. Include some sketches, if you think that will be helpful.

**Solution.** The characteristic equation is $mr^2 + 4r + 1 = 0$, which has roots $r = \dfrac{-4 \pm \sqrt{16 - 4m}}{2m}$.

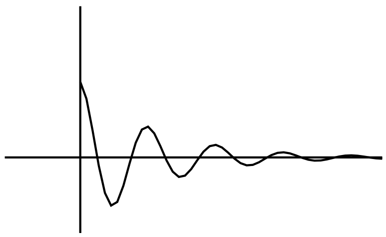
As long as $m < 4$, there are two real roots, so the system is overdamped. This means that the solution is a sum of decaying exponentials, and it looks something like this:



When $m$ is closer to 1, the decay to the equilibrium position is slower; it decays more quickly as $m$ approaches 4.

When $m$ reaches 4, there is only one root, and the system is critically damped. The general solution is $y = c_1 e^{-t/2} + c_2 t e^{-t/2}$, and the graph looks very similar to the decaying exponential in the previous case.

When $m$ is bigger than 4, the system is underdamped: the roots are complex, with negative real parts. So the solutions are decaying exponentials times trig functions – you get oscillations with decreasing amplitude:



The frequency of the oscillations is $\sqrt{4m-16}/2m$, which increases as $m$ goes from 4 to 8, and then starts decreasing.

4. Do EITHER (a) OR (b). Indicate clearly which one you are doing. You cannot get credit for both.

(a) Find the general solution of $y'' + 2y' + 5y = e^{-t}\sec 2t$.

**Solution.** I need to use variation of parameters to do this. First I'll find $y_h$. The characteristic equation is $r^2 + 2r + 5 = 0$, which has roots $r = -1 \pm 2i$. So $y_h = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$. From this, I extract two solutions to the homogeneous equation: $y_1 = e^{-t}\cos 2t$ and $y_2 = e^{-t}\sin 2t$.

In the variation of parameters formula, I need to know $W = y_1 y_2' - y_1' y_2$, so I'll compute that first:

$$\begin{aligned}
W &= y_1 y_2' - y_1' y_2 \\
&= (e^{-t}\cos 2t)(-e^{-t}\sin 2t + 2e^{-t}\cos 2t) - (-e^{-t}\cos 2t - 2e^{-t}\sin 2t)(e^{-t}\sin 2t) \\
&= -e^{-2t}\cos 2t \sin 2t + 2e^{-2t}\cos^2 2t + e^{-2t}\cos 2t \sin 2t + 2e^{-2t}\sin^2 2t \\
&= 2e^{-2t}(\cos^2 2t + \sin^2 2t) = 2e^{-2t}.
\end{aligned}$$

Now I'll find a particular solution:

$$\begin{aligned}
y_p &= -y_1 \int \frac{y_2 g(t)}{W} dt + y_2 \int \frac{y_1 g(t)}{W} dt \\
&= -e^{-t}\cos 2t \int \frac{e^{-t}\sin 2t e^{-t}\sec 2t}{2e^{-2t}} dt + e^{-t}\sin 2t \int \frac{e^{-t}\cos 2t e^{-t}\sec 2t}{2e^{-2t}} dt \\
&= -e^{-t}\cos 2t \int \frac{\sin 2t}{2\cos 2t} dt + e^{-t}\sin 2t \int \frac{\cos 2t}{2\cos 2t} dt \\
&= -\frac{1}{2} e^{-t}\cos 2t \int \tan 2t \, dt + \frac{1}{2} e^{-t}\sin 2t \int dt \\
&= -\frac{1}{4} e^{-t}\cos 2t \ln|\sec 2t| + \frac{1}{2} t e^{-t}\sin 2t.
\end{aligned}$$

So the general solution is

$$\boxed{y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t - \frac{1}{4} e^{-t}\cos 2t \ln|\sec 2t| + \frac{1}{2} t e^{-t}\sin 2t}.$$

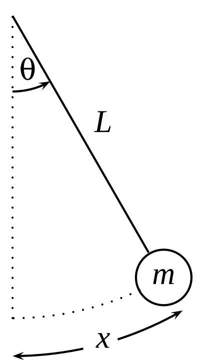
(b) One solution of $t^2y'' + 2ty' - 2y = 0$ is $y_1 = t$. Find the general solution.

**Solution.** I should use reduction of order here, so let $y_2 = vy_1 = vt$. Then $y_2' = v't + v$ and $y_2'' = v''t + 2v'$. Plug these into the equation:

$$t^2(v''t + 2v') + 2t(v't + v) - 2vt = 0,$$
$$t^3v'' + 4t^2v' = 0.$$

Now let $w = v'$; then this becomes $t^3w' + 4t^2w = 0$, or $w' = -4w/t$. This is separable: $dw/w = -4dt/t$. Integrate both sides: $\ln w = -4\ln t = \ln(t^{-4})$. Exponentiate both sides: $w = t^{-4}$. (I've left out the constant of integration here. If you include it, you will get $w = At^{-4}$.) Integrate $w$ to get $v$: $v = ct^{-3}$ for some constant $c$ ($c$ is $-1/3$, I suppose, but it doesn't actually matter). Since $y_2 = vt$, I now know that $y_2 = ct^{-2}$. So the general solution is $\boxed{y = c_1t + c_2t^{-2}}$.

5. (bonus problem) An object of mass $m$ is attached to the end of a massless rod of length $L$, forming a pendulum. Assume that there is no friction, and find a differential equation describing the motion of the mass. Explain your reasoning: don't just write down a bunch of equations. (I would suggest letting $\theta$ be the angle through which the object swings, and trying to find a differential equation involving $\theta$ and $t$.) According to your equation, does the motion of the object depend on the mass of the object? Does it depend on the length of the pendulum?




**Solution.** The basic differential equation here is, of course, $F = ma$. The object is moving along the arc of a circle, so let $x$ be the position of the object measured along that arc, as indicated in the picture. Then $x = L\theta$ (this is one of the virtues of radians – they make it easy to measure arc lengths). The velocity of the object is $x' = L\theta'$, and its acceleration is $x'' = L\theta''$. If I can find the force acting on the object, I'm in business. The only force is gravity, which is acting downward with a force of $-mg$. Since the object is constrained to move along the arc, the relevant part of the force is the component along the arc, which is $-mg\sin\theta$. So the differential equation is $ma = F$, which is

$$mL\theta'' = -mg\sin\theta.$$

I can cancel the $m$'s and rewrite this as

$$\boxed{\theta'' + \frac{g}{L}\sin\theta = 0}.$$

There is no $m$ here, so the motion does not depend on the mass. It does depend on $L$, though. Note also that this equation is not linear.