Math 136: Homework 3

Due Thursday, April 12

(1) Let V be the vector space of infinitely differentiable functions on the interval $[0,1]$. Show that the map $T : V \to V$ defined by

$$(Tf)(x) = \int_0^x f(t)\,dt$$

is a linear transformation. Prove that T is injective. Is T surjective? Explain your answer.

(2) Let V be the vector space of 2×2 matrices, let $A = \left(\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, and define a linear transformation L by

$$\begin{aligned} L : V &\to V, \\ B &\mapsto BA. \end{aligned}$$

Find a basis for the kernel of L.

(3) The following exercises show that the set $\mathbf{C}$ of complex numbers can be represented as a set of 2×2 matrices of a certain form.

(a) Show that the complex numbers $\mathbf{C}$ under addition and multiplication by real numbers can be viewed as a 2-dimensional vector space.

(b) Let $M(2)$ denote the space of 2×2 matrices, and let $L : \mathbf{C} \to M(2)$ be the map defined by $L(x + iy) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. Verify that L is a linear transformation. (That is, $L(z_1 + z_2) = L(z_1) + L(z_2)$ and $L(cz) = cL(z)$, for all $z_1, z_2, z \in \mathbf{C}$ and $c \in \mathbf{R}$.)

(c) Show that L satisfies the identity $L(z_1 z_2) = L(z_1)L(z_2)$ for all $z_1, z_2 \in \mathbf{C}$.

(d) What is the rank of L? What does this tell you about the kernel and image of L?

(4) Let A be a 3×3 upper triangular matrix:

$$A = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}.$$

(a) Prove that if $adf = 0$, then A is not invertible.

(b) Prove that if $adf \neq 0$, then A is invertible, and compute its inverse.