Math 136: Homework 4

Due Thursday, April 26

(1) Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$, and $P_3 = (x_3, y_3)$ be three non-collinear points in $\mathbb{R}^2$. Show that the equation of the circle passing through these points is given by the equation

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ 1 & x_3 & y_3 & x_3^2+y_3^2 \end{vmatrix} = 0.$$

(2) Let A be a real $n \times n$ matrix and consider the linear map $T_A : \mathbb{R}^n \to \mathbb{R}^n$ given by multiplication by A. Suppose that $\lambda = \alpha + i\beta \in \mathbb{C}$, $\beta \neq 0$, is a complex eigenvalue of A, with complex eigenvector $\mathbf{z} = \mathbf{x}_1 + i\mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are column vectors in $\mathbb{R}^n$ (not both zero). Thus,

$$A\mathbf{z} = (A\mathbf{x}_1) + i(A\mathbf{x}_2) = (\alpha\mathbf{x}_1 - \beta\mathbf{x}_2) + i(\alpha\mathbf{x}_2 + \beta\mathbf{x}_1).$$

(a) Prove that the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent and, therefore, span a 2-dimensional subspace $W = \text{span}(\mathbf{x}_1, \mathbf{x}_2) \subset \mathbb{R}^n$.

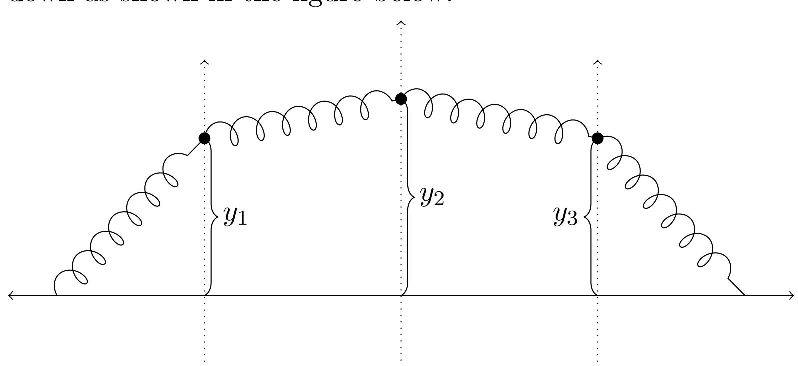
(b) Show that T_A restricts to define a linear map $T_W : W \to W$ and that the matrix of T_W with respect to the basis $(\mathbf{x}_1, \mathbf{x}_2)$ of W is

$$|\lambda| \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

where $\lambda = |\lambda| e^{i\theta}$ (the polar form of λ).

(c) What is the geometrical interpretation of the result of part (b)?

(3) Three objects of mass m connected by springs are free to move up and down as shown in the figure below.



Let y_k, for $k = 1, 2, 3$, denote the height of the kth object above the x-axis. Let F_k denote the vertical component of the net force on the kth object. One can show that if all the heights y_k are small, then

$$F_1 = -2Ky_1 + Ky_2\,, \quad F_2 = Ky_1 - 2Ky_2 + Ky_3\,, \quad F_3 = Ky_2 - 2Ky_3\,.$$

Let $\omega = \sqrt{K/m}$.

(a) Show that Newton's equations of motion can be written in the matrix form

$$\frac{d^2\mathbf{y}}{dt^2} + A\mathbf{y} = \mathbf{0},$$

where

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \text{ and } A = \omega^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(b) Find the general solution by finding eigenvalues and eigenvectors for the appropriate matrix.

(c) Interpret the eigenvectors you find.

(d) Suppose that at time $t = 0$, $y_1 = 1$, $y_2 = 0$, $y_3 = -1$, and the velocity of every mass is zero. Find $\mathbf{y}(t)$ for all t.