A-INFINITY STRUCTURE ON EXT-ALGEBRAS

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Abstract. Let A be a connected graded algebra and let E denote its Ext-algebra $\bigoplus_i \operatorname{Ext}^i_A(k_A, k_A)$. There is a natural A_∞-structure on E, and we prove that this structure is mainly determined by the relations of A. In particular, the coefficients of the A_∞-products m_n restricted to the tensor powers of $\operatorname{Ext}^1_A(k_A, k_A)$ give the coefficients of the relations of A. We also relate the m_n's to Massey products.

Introduction

The notions of A_∞-algebra and A_∞-space were introduced by Stasheff in the 1960s [St1]. Since then, more and more theories involving A_∞-structures (and its cousins, E_∞ and L_∞) have been discovered in several areas of mathematics and physics. Kontsevich's talk [Ko] at the ICM 1994 on categorical mirror symmetry has had an influence in developing this subject. The use of A_∞-algebras in noncommutative algebra and the representation theory of algebras was introduced by Keller [Ke1, Ke2, Ke3]. Recently the authors of this paper used the A_∞-structure on the Ext-algebra $\operatorname{Ext}^*_A(k, k)$ to study the non-Koszul Artin-Schelter regular algebras A of global dimension four [LP3]. The information about the higher multiplications on $\operatorname{Ext}^*_A(k, k)$ is essential and very effective for this work.

Throughout let k be a commutative base field. The definition of an A_∞-algebra will be given in Section 1. Roughly speaking, an A_∞-algebra is a graded vector space E equipped with a sequence of "multiplications" $(m_1, m_2, m_3, \cdots)$: m_1 is a differential, m_2 is the usual product, and the higher m_n's are homotopies which measure the degree of associativity of m_2. An associative algebra E (concentrated in degree 0) is an A_∞-algebra with multiplications $m_n = 0$ for all $n \neq 2$, so sometimes we write an associative algebra as (E, m_2). A differential graded (DG) algebra (E, d) has multiplication m_2 and derivation $m_1 = d$; this makes it into an A_∞-algebra with $m_n = 0$ for $n \geq 3$, and so it could be written as (E, m_1, m_2).

Let A be a connected graded algebra, and let k_A be the right trivial A-module $A/A_{\geq 1}$. The Ext-algebra $\bigoplus_{i \geq 0} \operatorname{Ext}^i_A(k_A, k_A)$ of A is the homology of a DG algebra, and hence by a theorem of Kadeishvili, it is equipped with an A_∞-algebra structure. Note that this A_∞-algebra structure is only unique up to A_∞-isomorphism, not on the nose. We use $\operatorname{Ext}^*_A(k_A, k_A)$ to denote both the usual associative Ext-algebra and the Ext-algebra with any choice for its A_∞-structure. By [LP1, Ex. 13.4] there is a graded algebra A such that the associative algebra $\operatorname{Ext}^*_A(k_A, k_A)$ does not contain enough information to recover the original algebra A; on the other hand, the information from the A_∞-algebra $\operatorname{Ext}^*_A(k_A, k_A)$ is sufficient to recover A.

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This is the point of Theorem A below, and this process of recovering the algebra from its Ext-algebra is one of the main tools used in [LP3].

We need some notation in order to state the theorem. We say that a graded vector space $V = \bigoplus V_i$ is *locally finite* if each V_i is finite-dimensional. We write the graded k-linear dual of V as $V^{\#}$. As our notation has so far indicated, we use subscripts to indicate the grading on A and related vector spaces. Also, the grading on A induces a bigrading on Ext. We write the usual, homological, grading with superscripts, and the second, induced, grading with subscripts.

Let $\mathfrak{m} = \bigoplus_{i\geq 1} A_i$ be the augmentation ideal of A. Let $Q = \mathfrak{m}/\mathfrak{m}^2$ be the graded vector space of generators of A. The relations in A naturally sit inside the tensor algebra on Q. In Section 4 we choose a vector space embedding of each graded piece A_s into the tensor algebra on Q: a map

$$A_s \hookrightarrow (\textstyle\bigoplus_{m\geq 1} Q^{\otimes m})_s,$$

which splits the multiplication map, and this choice affects how we choose the minimal generating set of relations. See Lemma 5.2 and the surrounding discussion for more details.

Theorem A. *Let A be a connected graded locally finite algebra. Let $Q = \mathfrak{m}/\mathfrak{m}^2$ be the graded vector space of generators of A. Let $R = \bigoplus_{s\geq 2} R_s$ be a minimal graded space of relations of A, with R_s chosen so that*

$$R_s \subset \textstyle\bigoplus_{1\leq i\leq s-1} Q_i \otimes A_{s-i} \subset (\bigoplus_{m\geq 2} Q^{\otimes m})_s.$$

For each $n \geq 2$ and $s \geq 2$, let $i_s : R_s \to (\bigoplus_{m\geq 2} Q^{\otimes m})_s$ be the inclusion map and let i_s^n be the composite

$$R_s \xrightarrow{i_s} (\textstyle\bigoplus_{m\geq 2} Q^{\otimes m})_s \to (Q^{\otimes n})_s.$$

Then there is a choice of A_∞-algebra structures $(m_2, m_3, m_4, \dots)$ on $E = \mathrm{Ext}^*_A(k_A, k_A)$ so that in any degree $-s$, the multiplication m_n of E restricted to $(E^1)^{\otimes n}_{-s}$ is equal to the map

$$(i_s^n)^{\#} : \big((E^1)^{\otimes n}\big)_{-s} = \big((Q^{\otimes n})_s\big)^{\#} \longrightarrow R_s^{\#} \subset E^2_{-s}.$$

In plain English, the multiplication maps m_n on classes in $\mathrm{Ext}^1_A(k_A, k_A)$ are determined by the relations in the algebra A.

Note that the space Q of generators need not be finite-dimensional – it only has to be finite-dimensional in each grading. Thus the theorem applies to infinitely generated algebras like the Steenrod algebra.

The authors originally announced the result in the following special case; this was used heavily in [LP3].

Corollary B (Keller's higher-multiplication theorem in the connected graded case)**.** *Let A be a connected graded algebra, finitely generated in degree 1. Let $R = \bigoplus_{n\geq 2} R_n$ be a minimal graded space of relations of A, chosen so that $R_n \subset A_1 \otimes A_{n-1} \subset A_1^{\otimes n}$. For each $n \geq 2$, let $i_n : R_n \to A_1^{\otimes n}$ be the inclusion map and let $i_n^{\#}$ be its k-linear dual. Then there is a choice of A_∞-algebra structures $(m_2, m_3, m_4, \dots)$ on $E = \mathrm{Ext}^*_A(k_A, k_A)$ so that the multiplication map m_n of E restricted to $(E^1)^{\otimes n}$ is equal to the map*

$$i_n^{\#} : (E^1)^{\otimes n} = (A_1^{\#})^{\otimes n} \longrightarrow R_n^{\#} \subset E^2.$$

Keller has the same result for a different class of algebras; indeed, his result was the inspiration for Theorem A. His result applies to algebras the form $k\Delta/I$ where Δ is a finite quiver and I is an admissible ideal of $k\Delta$; this was stated in [Ke4, Proposition 2] without proof. This class of algebras includes those in Corollary B, but since the algebra A in Theorem A need not be finitely generated, that theorem is not a special case of Keller's result. A version of Corollary B was also proved in a recent paper by He and Lu [HL] for $\mathbb{N}$-graded algebras $A = A_0 \oplus A_1 \oplus \cdots$ with $A_0 = k^{\oplus n}$ for some $n \geq 1$, and which are finitely generated by $A_0 \oplus A_1$. Their proof was based on the one here (see [HL, page 356]).

Here is an outline of the paper. We review the definitions of A_∞-algebras and Adams grading in Section 1. In Section 2 we discuss Kadeishvili's and Merkulov's results about the A_∞-structure on the homology of a DG algebra. In Section 3 we use Merkulov's construction to show that the A_∞-multiplication maps m_n compute Massey products, up to a sign – see Theorem 3.1 and Corollary A.5 for details. In Section 4 the bar construction is described: the dual of the bar construction is a DG algebra whose homology is Ext, and so leads to an A_∞-structure on Ext algebras. Then we give a proof of Theorem A in Section 5, and in Section 6 we give a few examples. Finally, there is an appendix in which we prove that Merkulov's construction is in some sense ubiquitous among A_∞-algebras; this appendix was inspired by a comment from the referee.

This paper began as an appendix in [LP3].

1. Definitions

In this section we review the definition of an A_∞-algebra and discuss grading systems. Other basic material about A_∞-algebras can be found in Keller's paper [Ke3]. Some examples of A_∞-algebras related to ring theory were given in [LP1]. Here is Stasheff's definition.

Definition 1.1. [St1, Part II, Definition 2.1] An A_∞*-algebra* over a base field k is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{p \in \mathbb{Z}} A^p$$

endowed with a family of graded k-linear maps

$$m_n : A^{\otimes n} \to A, \quad n \geq 1,$$

of degree $2 - n$ satisfying the following *Stasheff identities*:

$$\textbf{SI(n)} \qquad \sum (-1)^{r+st} m_u(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = 0$$

for all $n \geq 1$, where the sum runs over all decompositions $n = r + s + t$ ($r, t \geq 0$ and $s \geq 1$), and where $u = r + 1 + t$. Here, id denotes the identity map of A. Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule. Some authors also use the terminology *strongly homotopy associative algebra* (or *sha algebra*) for A_∞-algebra.

The degree of m_1 is 1 and the identity **SI(1)** is $m_1 m_1 = 0$. This says that m_1 is a differential on A. The identity **SI(2)** is

$$m_1 m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1)$$

as maps $A^{\otimes 2} \to A$. So the differential m_1 is a graded derivation with respect to m_2. Note that m_2 plays the role of multiplication although it may not be associative.

The degree of m_2 is zero. The identity **SI(3)** is

$$m_2(\mathrm{id}\otimes m_2 - m_2\otimes \mathrm{id}) = m_1m_3 + m_3(m_1\otimes \mathrm{id}\otimes \mathrm{id} + \mathrm{id}\otimes m_1\otimes \mathrm{id} + \mathrm{id}\otimes \mathrm{id}\otimes m_1)$$

as maps $A^{\otimes 3}\to A$. If either m_1 or m_3 is zero, then m_2 is associative. In general, m_2 is associative up to the chain homotopy given by m_3.

When $n\geq 3$, the map m_n is called a *higher multiplication*. We write an A_∞-algebra A as $(A, m_1, m_2, m_3, \cdots)$ to indicate the multiplications m_n. We also assume that every A_∞-algebra in this paper contains an identity element 1 with respect to the multiplication m_2 that satisfies the following *strictly unital condition*:

$$\text{If } n\neq 2 \text{ and } a_i = 1 \text{ for some } i, \text{ then } m_n(a_1\otimes\cdots\otimes a_n) = 0.$$

In this case, 1 is called the *strict unit* or *identity* of A.

We are mainly interested in graded algebras and their Ext-algebras. The grading appearing in a graded algebra may be different from the grading appearing in the definition of the A_∞-algebra. We introduce the Adams grading for an A_∞-algebra, as follows. Let G be an abelian group. (In this paper, G will always be free abelian of finite rank.) Consider a bigraded vector space

$$A = \bigoplus\nolimits_{p\in\mathbb{Z}, i\in G} A_i^p$$

where the upper index p is the grading appearing in Definition 1.1, and the lower index i is an extra grading, called the *G-Adams grading*, or *Adams grading* if G is understood. We also write

$$A^p = \bigoplus\nolimits_{i\in G} A_i^p \quad\text{and}\quad A_i = \bigoplus\nolimits_{p\in\mathbb{Z}} A_i^p.$$

The degree of a nonzero element in A_i^p is (p, i), and the second degree is called the *Adams degree*. For an A_∞-algebra A to have an Adams grading, the map m_n in Definition 1.1 must be of degree $(2-n, 0)$: each m_n must preserve the Adams grading. When A is an associative G-graded algebra $A = \bigoplus_{i\in G} A_i$, we view A as an A_∞-algebra (or a DG algebra) concentrated in degree 0, viewing the given grading on A as the Adams grading. The Ext-algebra of a graded algebra is bigraded; the grading inherited from the graded algebra is the Adams grading, and we keep using the lower index to denote the Adams degree.

Assume now that $G = \mathbb{Z}$, since we are mainly interested in this case. Write

$$A^{\geq n} = \bigoplus\nolimits_{p\geq n} A^p \quad\text{and}\quad A_{\geq n} = \bigoplus\nolimits_{i\geq n} A_i,$$

and similarly for $A^{\leq n}$ and $A_{\leq n}$. An A_∞-algebra A with a $\mathbb{Z}$-Adams grading is called *Adams connected* if (a) $A_0 = k$, (b) $A = A_{\geq 0}$ or $A = A_{\leq 0}$, and (c) A_i is finite-dimensional for all i. When $G = \mathbb{Z}\times G_0$, we define *Adams connected* in the same way after omitting the G_0-grading. If A is a connected graded algebra which is finite-dimensional in each degree, then it is Adams connected when viewed as an A_∞-algebra concentrated in degree 0.

The following result is a consequence of Theorem A, and it will be proved at the end of Section 5. There might be several quasi-isomorphic A_∞-structures on $E := \mathrm{Ext}_A^*(k_A, k_A)$; we call these different structures *models* for the quasi-isomorphism class of E.

Proposition 1.2. *Let A be a $\mathbb{Z}\oplus G$-Adams graded algebra, such that with respect to the $\mathbb{Z}$-grading, A is locally finite. Then there is an A_∞-model for E such that the multiplications m_n in Theorem A preserve the $\mathbb{Z}\oplus G$ grading.*

2. Kadeishvili's theorem and Merkulov's construction

Let A and B be two A_∞-algebras. A *morphism* of A_∞-algebras $f : A \to B$ is a family of k-linear graded maps

$$f_n : A^{\otimes n} \to B, \quad n \geq 1,$$

of degree $1-n$ satisfying the following *Stasheff morphism identities*:

MI(n) $$\sum (-1)^{r+st} f_u(\mathrm{id}^{\otimes r} \otimes m_s \otimes \mathrm{id}^{\otimes t}) = \sum (-1)^w m_q(f_{i_1} \otimes f_{i_2} \otimes \cdots \otimes f_{i_q})$$

for all $n \geq 1$, where the first sum runs over all decompositions $n = r+s+t$ with $s \geq 1$ and $r, t \geq 0$, where $u = r+1+t$, and the second sum runs over all $1 \leq q \leq n$ and all decompositions $n = i_1 + \cdots + i_q$ with all $i_s \geq 1$. The sign on the right-hand side is given by

$$w = (q-1)(i_1-1) + (q-2)(i_2-1) + \cdots + 2(i_{q-2}-1) + (i_{q-1}-1).$$

When A and B have a strict unit (as we always assume), an A_∞-morphism is also required to satisfy the following extra *unital morphism conditions*:

$$f_1(1_A) = 1_B$$

where 1_A and 1_B are the strict units of A and B respectively, and

$$f_n(a_1 \otimes \cdots \otimes a_n) = 0$$

if $n \geq 2$ and $a_i = 1_A$ for some i.

If A and B have Adams gradings indexed by the same group, then the maps f_i are required to preserve the Adams degree.

A morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. A morphism is *strict* if $f_i = 0$ for all $i \neq 1$. The *identity morphism* is the strict morphism f such that f_1 is the identity of A. When f is a strict morphism from A to B, then the identity **MI(n)** becomes

$$f_1 m_n = m_n(f_1 \otimes \cdots \otimes f_1).$$

A morphism $f = (f_i)$ is called a *strict isomorphism* if it is strict with f_1 a vector space isomorphism.

Let A be an A_∞-algebra. Its cohomology ring is defined to be

$$HA := \ker m_1 / \operatorname{im} m_1.$$

The map $m_2 : A \otimes A \to A$ induces an associative multiplication on HA. The following result, due to Kadeishvili [Ka], is a basic and important property of A_∞-algebras.

Theorem 2.1. [Ka] *Let A be an A_∞-algebra and let HA be the cohomology ring of A. There is an A_∞-algebra structure on HA with $m_1 = 0$, m_2 induced by the multiplication on A, and the higher multiplications constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $HA \to A$ lifting the identity of HA. This A_∞-algebra structure on HA is unique up to A_∞-isomorphism.*

Kadeishvili's construction is very general. We would like to describe some specific A_∞-structures that we can work with. Merkulov constructed a special class of higher multiplications for HA in [Me], in which the higher multiplications can be defined inductively; this way, the A_∞-structure can be described more explicitly,

and hence used more effectively. For our purposes we will describe a special case of Merkulov's construction, assuming that A is a DG algebra.

Let A be a DG algebra with differential ∂ and multiplication $\cdot$. Denote by B^n and Z^n the coboundaries and cocycles of A^n, respectively. Then there are subspaces H^n and L^n such that

$$Z^n = B^n \oplus H^n$$

and

$$A^n = Z^n \oplus L^n = B^n \oplus H^n \oplus L^n. \tag{2.1.1}$$

We will identify HA with $\bigoplus_n H^n$, or embed HA into A by cocycle-sections $H^n \subset A^n$. *There are many different choices of H^n and L^n.*

Note that if A has an Adams grading, then the decompositions above will be chosen to respect the Adams grading, and all maps constructed below will preserve the Adams grading.

Let $p = Pr_H : A \to A$ be the projection to $H := \bigoplus_n H^n$, and let $G : A \to A$ be a homotopy from id_A to p. Hence we have $\mathrm{id}_A - p = \partial G + G\partial$. The map G is not unique, and we want to choose G carefully, so we define it as follows: for every n, $G^n : A^n \to A^{n-1}$ is the map which satisfies

- $G^n = 0$ when restricted to L^n and H^n, and
- $G^n = (\partial^{n-1}|_{L^{n-1}})^{-1}$ when restricted to B^n.

So the image of G^n is L^{n-1}. It follows that $G^{n+1}\partial^n = Pr_{L^n}$ and $\partial^{n-1}G^n = Pr_{B^n}$.

Define a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$ of degree $2-n$ as follows. There is no map λ_1, but we formally define the "composite" $G\lambda_1$ by $G\lambda_1 = -\mathrm{id}_A$. λ_2 is the multiplication of A, namely, $\lambda_2(a_1 \otimes a_2) = a_1 \cdot a_2$. For $n \geq 3$, λ_n is defined by the recursive formula

$$\lambda_n = \sum_{\substack{s+t=n,\\ s,t\geq 1}} (-1)^{s+1}\lambda_2[G\lambda_s \otimes G\lambda_t]. \tag{2.1.2}$$

We abuse notation slightly, and use p to denote both the map $A \to A$ and also (since the image of p is HA) the map $A \to HA$; we also use λ_i both for the map $A^{\otimes i} \to A$ and for its restriction to $(HA)^{\otimes i}$: that is, the map $(HA)^{\otimes i} \to A$,

Merkulov reproved Kadeishvili's result in [Me].

Theorem 2.2. [Me] *Let $m_i = p\lambda_i$. Then $(HA, m_2, m_3, \cdots)$ is an A_∞-algebra.*

We can also display the quasi-isomorphism between HA and A directly.

Proposition 2.3. *Let $\{\lambda_n\}$ be defined as above. For $i \geq 1$ let $f_i = -G\lambda_i : (HA)^{\otimes i} \to A$, and for $i \geq 2$ let $m_i = p\lambda_i : (HA)^{\otimes i} \to HA$. Then $(HA, m_2, m_3, \cdots)$ is an A_∞-algebra and $f := \{f_i\}$ is a quasi-isomorphism of A_∞-algebras.*

Proof. This construction of $\{m_i\}$ and $\{f_i\}$ is a special case of Kadeishvili's construction. □

Note that if A has an Adams grading, then by construction all maps m_i and f_i preserve the Adams degree.

Definition 2.4. Any A_∞-algebra constructed as in Theorem 2.2 and Proposition 2.3 is called a *Merkulov model* of A, and may be denoted by $H_{\mathrm{Mer}}A$.

The particular model depends on the decomposition (2.1.1), but all Merkulov models of A are quasi-isomorphic to each other. Also, although this construction may seem rather specific, it is actually quite general: in the appendix, we show that any A_∞-algebra H with $m_1 = 0$ may be obtained from a Merkulov model of a DG algebra. See Theorem A.4 for details.

Next we consider the unital condition.

Lemma 2.5. *Suppose H^0 is chosen to contain the unit element of A. Then $H_{\mathrm{Mer}}A$ satisfies the strictly unital condition, and the morphism $f = \{f_i\}$ satisfies the unital morphism conditions.*

Proof. First of all, $1 \in H^0$ is a unit with respect to m_2. We use induction on n to show the following, for $n \geq 3$:

$(a)_n$: $f_{n-1}(a_1 \otimes \cdots \otimes a_{n-1}) = 0$ if $a_i = 1$ for some i.

$(b)_n$: $\lambda_n(a_1 \otimes \cdots \otimes a_n) \in L := \bigoplus_n L^n$ if $a_i = 1$ for some i.

$(c)_n$: $m_n(a_1 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some i.

The strictly unital condition is $(c)_n$. The unital morphism condition is $(a)_n$.

We first prove $(a)_3$. For $a \in H$,

$$f_2(1 \otimes a) = -G\lambda_2(1 \otimes a) = -G(a) = 0,$$

since $G|_H = 0$. Similarly, $f_2(a \otimes 1) = 0$. This proves $(a)_3$. Now suppose for some $n \geq 3$ that $(a)_i$ holds for all $3 \leq i \leq n$. By definition,

$$\lambda_n = \sum_{s=1}^{n-1} (-1)^{s+1} \lambda_2(f_s \otimes f_{n-s}).$$

If $a_1 = 1$, $(a)_n$ implies that

$$\lambda_n(a_1 \otimes \cdots \otimes a_n) = f_{n-1}(a_2 \otimes \cdots \otimes a_n) \in L.$$

Similarly, if $a_n = 1$, we have $\lambda_n(a_1 \otimes \cdots \otimes a_n) \in L$. If $a_i = 1$ for $1 < i < n$, then $\lambda_n(a_1 \otimes \cdots \otimes a_n) = 0$. Therefore $(a)_i$ for $i \leq n$ implies $(b)_n$. Since $p(L) = 0$, $(c)_n$ follows from $(b)_n$. Since $G(L) = 0$, $(a)_{n+1}$ follows from $(b)_n$. Induction completes the proof. □

Lemma 2.6. *Let (A, ∂) be a DG algebra and let $e \in A^0$ be an idempotent such that $\partial(e) = 0$. Let $D = eAe$ and $C = (1-e)A + A(1-e)$.*

(a) *If $HC = 0$, then we can choose Merkulov models so that $H_{\mathrm{Mer}}A$ is strictly isomorphic to $H_{\mathrm{Mer}}D$. As a consequence A and D are quasi-isomorphic as A_∞-algebras.*

(b) *If moreover HA is Adams connected, then $H^0_{\mathrm{Mer}}A$ and $H^0_{\mathrm{Mer}}D$ in part (a) can be chosen to contain the unit element.*

Proof. First of all, D is a sub-DG algebra of A with identity e. Since $A = D \oplus C$ as chain complexes, the group of coboundaries B^n decomposes as $B^n = B^n_D \oplus B^n_C$, where $B^n_D = B^n \cap D$ and $B^n_C = B^n \cap C$. Since $HC = 0$, we can choose H and L so that they decompose similarly (with $H_C = 0$), giving the following direct sum decompositions:

$$A^n = D^n \oplus C^n = (B^n_D \oplus H^n_D \oplus L^n_D) \oplus (B^n_C \oplus L^n_C),$$
$$A^n = B^n \oplus H^n \oplus L^n = (B^n_D \oplus B^n_C) \oplus H^n_D \oplus (L^n_D \oplus L^n_C).$$

It follows from the construction before Theorem 2.2 that $H_{\mathrm{Mer}}A = H_{\mathrm{Mer}}D$. We choose H^0_D to contain e. By Lemma 2.5, e is the strict unit of $H_{\mathrm{Mer}}D$; hence e is the strict unit of $H_{\mathrm{Mer}}A$, but note that the unit 1 of A may not be in HA.

Now suppose HA is Adams connected with unit u. Let $H^0 = ku \oplus H^0_{\geq 1}$ (or $H^0 = ku \oplus H^0_{\leq -1}$ if negatively connected graded). Replace H^0 by $k1 \oplus H^0_{\geq 1}$ and keep the other subspaces B^n, H^n, and L^n the same. Let $\overline{H_{\mathrm{Mer}}A}$ denote the new Merkulov model with the new choice of H^0. Then by Lemma 2.5, 1 is the strict unit of $\overline{H_{\mathrm{Mer}}A}$. By construction, we have $(\overline{H_{\mathrm{Mer}}A})_{\geq 1} = (H_{\mathrm{Mer}}A)_{\geq 1}$ as A_∞-algebras without unit. By the unital condition, we see that $\overline{H_{\mathrm{Mer}}A}$ is strictly isomorphic to $H_{\mathrm{Mer}}A$. □

3. Massey products

It is common to view A_∞-algebras as algebras which are strongly homotopy associative: not associative on the nose, but associative up to all higher homotopies, as given by the m_n's. Any A_∞-algebra in which the differential m_1 is zero, such as the cohomology of a DG algebra, is strictly associative, though, and in such a case, it is natural to wonder about the role of the higher multiplications. On the other hand, the cohomology of a DG algebra is the natural setting for Massey products. With Merkulov's construction in hand, we give a proof of a folk theorem which connects the higher multiplication maps with Massey products: we prove that they are essentially the same, up to a sign. We start by reviewing Massey products. We use the sign conventions from May [Ma]; see also Ravenel [Ra, A1.4].

If a is an element of a DG algebra A, we write $\overline{a}$ for $(-1)^{1+\deg a}a$. (This notation helps to simplify some formulas.)

The length two Massey product $\langle \alpha_1, \alpha_2 \rangle$ is the ordinary product: $\langle \alpha_1, \alpha_2 \rangle = \alpha_1\alpha_2$. (For consistency with the higher products, one could also define $\langle \alpha_1, \alpha_2 \rangle$ as being the set $\{\alpha_1\alpha_2\}$, but we do not take this point of view.)

The Massey triple product is defined as follows: suppose given classes $\alpha_1, \alpha_2, \alpha_3 \in HA$ which are represented by cocycles $a_{01}, a_{12}, a_{23} \in A$, respectively, and suppose that $\alpha_1\alpha_2 = 0 = \alpha_2\alpha_3$. Then there are cochains a_{02} and a_{13} so that $\partial(a_{02}) = \overline{a}_{01}a_{12}$ and $\partial(a_{13}) = \overline{a}_{12}a_{23}$, and so

$$\overline{a}_{02}a_{23} + \overline{a}_{01}a_{13}$$

is a cocycle and represents a cohomology class. One can choose different cochains for a_{02} and a_{13}: one can replace a_{02} with $a_{02} + z$ for any cocycle z, for instance, and this can produce a different cohomology class. The *length 3 Massey product* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is the set of cohomology classes which arise from all such choices of a_{02} and a_{13}.

More generally, one may consider the length n Massey product for any $n \geq 3$. Suppose that we have cohomology classes α_i for $1 \leq i \leq n$, and suppose that whenever $i < j$ and $j - i < n - 1$, each length $j - i + 1$ Massey product $\langle \alpha_i, \ldots, \alpha_j \rangle$ is defined and contains zero. Then the *length n Massey product* $\langle \alpha_1, \ldots, \alpha_n \rangle$ exists and is defined as follows: for all i and j with $0 < i < j < n$ and $j - i + 1 \leq n$, one finds cochains a_{ij} as follows: $a_{i-1,i}$ is a cocycle representing the cohomology class α_i, and for $j > i + 1$, a_{ij} is a cocycle satisfying

$$\partial(a_{ij}) = \sum_{i<k<j} \overline{a}_{ik}a_{kj}.$$

Such a collection $\{a_{ij}\}$ is called a *defining system* for the Massey product, and $\langle \alpha_1, \dots, \alpha_n \rangle$ is the set of cohomology classes represented by cocycles of the form $\sum_{0<i<n} \overline{a}_{0i} a_{in}$, for all defining systems $\{a_{ij}\}$. (It is tedious but straightforward to check that each such sum is a cocycle.) One can see that $\langle \alpha_1, \dots, \alpha_n \rangle \subset H^{s-(n-2)} A$, where s is the sum of the degrees of the α_i's, which means that $\langle \alpha_1, \dots, \alpha_n \rangle$ is in the same degree as $m_n(\alpha_1 \otimes \cdots \otimes \alpha_n)$. This is not a coincidence.

Theorem 3.1. *Let A be a DG algebra. Up to a sign, the higher A_∞-multiplications on HA give Massey products. More precisely, suppose that HA is given an A_∞-algebra structure by Merkulov's construction, and fix $n \geq 3$. If $\alpha_1, \dots, \alpha_n \in HA$ are elements such that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined, then*

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \dots, \alpha_n \rangle,$$

where

$$b = 1 + \deg \alpha_{n-1} + \deg \alpha_{n-3} + \deg \alpha_{n-5} + \cdots .$$

In Corollary A.5 below, we show that this result depends only on the A_∞-algebra structure of HA up to isomorphism – it holds even if the structure does not come from Merkulov's construction.

The authors have been unable to find an account of this theorem in its full generality, but for some related results, see [Ka, p. 233], [St2, Chapter 12], and [JL, 6.3–6.4].

Now, there are choices made in Merkulov's construction – in particular, the choices of the splittings (2.1.1) – and different choices can lead to different elements in the Massey products, as well as different (but quasi-isomorphic) A_∞-algebra structures. See Example 6.5 for an example of a DG algebra whose homology has a Massey product $\langle \alpha, \beta, \gamma \rangle$ containing several elements, such that for each element ζ in that Massey product, one may choose a Merkulov model such that $(-1)^b m_3(\alpha \otimes \beta \otimes \gamma) = \zeta$.

In any case, any choice of A_∞-structure via Merkulov's construction gives a set of choices for an element of each Massey product which is in "coherent" in the sense that the A_∞-algebra structure must satisfy the Stasheff identities. Of course, the A_∞-multiplications are also universally defined, not just when certain products are zero.

Proof. The proof is by induction on n.

The theorem discusses the situation when $n \geq 3$, but we will also use the formula when $n = 2$ in the induction: when $n = 2$, we have

$$m_2(\alpha_1 \otimes \alpha_2) = \alpha_1 \alpha_2 = \langle \alpha_1, \alpha_2 \rangle = (-1)^{1+\deg \alpha_1} \overline{\alpha}_1 \alpha_2.$$

Now let $n = 3$. We use Merkulov's construction for the A_∞-algebra structure on HA, so we choose splittings as in (2.1.1), and we define the multiplication maps m_n as in Theorem 2.2. We use a little care when choosing the elements $a_{ij} \in A$: we define a_{02} by $G(\overline{a}_{01} a_{12}) = a_{02}$, so $\partial(a_{02}) = \overline{a}_{01} a_{12}$ and $G\lambda_2(\alpha_1 \otimes \alpha_2) = (-1)^{1+\deg \alpha_1} a_{02}$. We define a_{13} similarly. Then we have

$$\begin{aligned} m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) &= p\lambda_2(G\lambda_1 \otimes G\lambda_2 - G\lambda_2 \otimes G\lambda_1)(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \\ &= p\left((-1)^{1+\deg \alpha_1 + 1 + \deg \alpha_2} a_{01} a_{13} + (-1)^{1+1+1+\deg \alpha_1} a_{02} a_{23}\right) \\ &= p\left((-1)^{1+\deg \alpha_2} \overline{a}_{01} a_{13} + (-1)^{1+\deg \alpha_2} \overline{a}_{02} a_{23}\right) \\ &= (-1)^{1+\deg \alpha_2} p\left(\overline{a}_{01} a_{13} + \overline{a}_{02} a_{23}\right). \end{aligned}$$

(Some signs here are due to the Koszul sign convention; for example, the map $G\lambda_2$ has degree 1, so $(G\lambda_1 \otimes G\lambda_2)(a_{01} \otimes a_{12} \otimes a_{23}) = (-1)^{\deg \alpha_1} G\lambda_1(a_{01}) \otimes G\lambda_2(a_{12} \otimes a_{23})$.)

The map p is the projection map from A to its summand H. Loosely, for any cocycle z, $p(z)$ is the cohomology class represented by z; more precisely, $p(z)$ is the unique class in $H \subset A$ which is cohomologous to z. In the situation here, the term in parentheses is a cocycle whose cohomology class is in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, so we get the desired result.

Assume that the result is true for m_i with $i < n$. Therefore for all $i < j$ with $j - i < n - 1$, we may choose elements $a_{i-1,j}$ by the formula

$$G\lambda_{j-i+1}(\alpha_i \otimes \cdots \otimes \alpha_j) = (-1)^{1+\deg \alpha_{j-1} + \deg \alpha_{j-3} + \cdots} a_{i-1,j}.$$

We write b_{ij} for the exponent of -1 here:

$$b_{ij} = 1 + \deg \alpha_{j-1} + \deg \alpha_{j-3} + \cdots .$$

The last term in this sum is $\deg \alpha_{j-(2k+1)}$, where k is the maximum such that $j - (2k+1) \geq i$. Note also for use with the Koszul sign convention that $G\lambda_i$ has degree $1 - i$. Then

$$\begin{aligned}
m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) &= p\lambda_2 \left(\sum_{s=1}^{n-1} (-1)^{s+1} G\lambda_s \otimes G\lambda_{n-s} \right) (a_{01} \otimes \cdots \otimes a_{n-1,n}) \\
&= p\Bigg(\sum_{s=1}^{n-1} (-1)^{s+1+(1-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s)} \\
&\qquad\qquad G\lambda_s(\alpha_1 \otimes \cdots \otimes \alpha_s) G\lambda_{n-s}(\alpha_{s+1} \otimes \cdots \otimes \alpha_n) \Bigg) \\
&= p\left(\sum_{s=1}^{n-1} (-1)^{s+1+(1-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s) + b_{1s} + b_{s+1,n}} \ a_{0s} a_{sn} \right) \\
&= p\Bigg(\sum_{s=1}^{n-1} (-1)^{s+1+(1-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s) + b_{1s} + b_{s+1,n}} \\
&\qquad\qquad (-1)^{1-s+1+\deg \alpha_1 + \cdots + \deg \alpha_s} \ \overline{a}_{0s} a_{sn} \Bigg) \\
&= p\left(\sum_{s=1}^{n-1} (-1)^{1+(-n+s)(\deg \alpha_1 + \cdots + \deg \alpha_s) + b_{1s} + b_{s+1,n}} \ \overline{a}_{0s} a_{sn} \right) \\
&= p\left(\sum_{s=1}^{n-1} (-1)^{b_{1n}} \ \overline{a}_{0s} a_{sn} \right),
\end{aligned}$$

where $b_{1n} = b$ is the sign as in the theorem: if $n - s$ is even, then the sign is $(-1)^{1+b_{1s}+b_{s+1,n}}$, and with $n - s$ even, we have $b_{1,n} = 1 + b_{1s} + b_{s+1,n}$. If $n - s$ is odd, then the sign is

$$(-1)^{1+b_{s+1,n}+1+\deg \alpha_s + \deg \alpha_{s-2} + \deg \alpha_{s-4} + \cdots} = (-1)^{b_{1n}},$$

as claimed. As with the $n = 3$ case, since the sum $\sum \overline{a}_{0s} a_{sn}$ is a cocycle, p sends it to the cohomology class that it represents, which is an element of the Massey product $\langle \alpha_1, \ldots, \alpha_n \rangle$. This finishes the proof. □

See Section 6 for some examples.

4. The bar construction and Ext

The bar/cobar construction is one of the basic tools in homological algebra. Everything in this section is well-known – see [FHT], for example – but we need the details for the proof in the next section.

Let A be a connected graded algebra and let k be the trivial A-module. Of course, the i-th Ext-group $\mathrm{Ext}^i_A(k_A, k_A)$ can be computed by the i-th cohomology of the complex $\mathrm{Hom}_A(P_A, k_A)$ where P_A is any projective (or free) resolution of k_A. Since P_A is projective, $\mathrm{Hom}_A(P_A, k_A)$ is quasi-isomorphic to $\mathrm{Hom}_A(P_A, P_A) = \mathrm{End}_A(P_A)$; hence $\mathrm{Ext}^i_A(k_A, k_A) \cong H^i(\mathrm{End}_A(P_A))$. Since $\mathrm{End}_A(P_A)$ is a DG algebra, the graded vector space $\mathrm{Ext}^*_A(k_A, k_A) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^i_A(k_A, k_A)$ has a natural algebra structure, and it also has an A_∞-structure by Kadeishvili's result Theorem 2.1. By [Ad, Chap.2], the Ext-algebra of a graded algebra A can also be computed by using the bar construction on A, which has several advantages over $\mathrm{End}_A(P_A)$: it tends to be somewhat smaller, it is connected, and it is natural in A.

First we review the shift functor. Let (M, ∂) be a complex with differential ∂ of degree 1, and let n be an integer. The nth shift of M, denoted by $S^n(M)$, is defined by

$$S^n(M)^i = M^{i+n}$$

and the differential of $S^n(M)$ is

$$\partial_{S^n(M)}(m) = (-1)^n \partial(m)$$

for all $m \in M$. If $f : M \to N$ is a homomorphism of degree p, then $S^n(f) : S^n(M) \to S^n(N)$ is defined by the formula

$$S^n(f)(m) = (-1)^{pn} f(m)$$

for all $m \in S^n(M)$. The functor S^n is an automorphism of the category of complexes.

The following definition is essentially standard, although sign conventions may vary; we use the conventions from [FHT, Sect.19]. Let A be an augmented DG algebra with augmentation $\epsilon : A \to k$, viewing k as a trivial DG algebra. Let $\mathfrak{m}$ be the kernel of ϵ and $S\mathfrak{m}$ be the shift of $\mathfrak{m}$. The *tensor coalgebra* on $S\mathfrak{m}$ is

$$T(S\mathfrak{m}) = k \oplus S\mathfrak{m} \oplus (S\mathfrak{m})^{\otimes 2} \oplus (S\mathfrak{m})^{\otimes 3} \oplus \cdots ,$$

where an element $Sa_1 \otimes Sa_2 \otimes \cdots \otimes Sa_n$ in $(S\mathfrak{m})^{\otimes n}$ is written as

$$[a_1|a_2|\cdots|a_n]$$

for $a_i \in \mathfrak{m}$, and with a comultiplication given by the formula

$$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

The degree of $[a_1|\cdots|a_n]$ is $\sum_{i=1}^n (\deg a_i - 1)$.

Definition 4.1. Let (A, ∂_A) be an augmented DG algebra and let $\mathfrak{m}$ denote the augmentation ideal $\ker(A \to k)$. The *bar construction* on A is the coaugmented differential graded coalgebra (DG coalgebra, for short) BA defined as follows:

- As a coaugmented graded coalgebra, BA is the tensor coalgebra $T(S\mathfrak{m})$.

- The differential in BA is the sum $d = d_0 + d_1$ of the coderivations given by

$$d_0([a_1|\cdots|a_m]) = -\sum_{i=1}^{m}(-1)^{n_i}[a_1|\cdots|\partial_A(a_i)|\cdots|a_m]$$

and

$$d_1([a_1]) = 0,$$
$$d_1([a_1|\cdots|a_m]) = \sum_{i=2}^{m}(-1)^{n_i}[a_1|\cdots|a_{i-1}a_i|\cdots|a_m] \qquad \text{if } m \geq 2,$$

where $n_i = \sum_{j<i}(\deg a_j - 1) = \deg[a_1|\cdots|a_{i-1}]$.

The cobar construction ΩC on a coaugmented DG coalgebra C is defined dually [FHT, Sect.19]. We omit the definition here since it is used only in two places, one of which is between Lemma 5.3 and Lemma 5.4, and the other is in Lemma 5.5. We provide details in the appendix – see Definition A.2.

In the rest of this section we assume that A is an augmented associative algebra. In this case $S\mathfrak{m}$ is concentrated in degree -1; hence the degree of $[a_1|\cdots|a_m]$ is $-m$. This means that the bar construction BA is graded by the *negative* of tensor length. The degree of the differential d is 1. We may think of BA as a complex with $(-i)$th term equal to $\mathfrak{m}^{\otimes i}$, the differential d mapping $\mathfrak{m}^{\otimes i}$ to $\mathfrak{m}^{\otimes i-1}$. If A has an Adams grading, denoted adeg, then BA has a bigrading that is defined by

$$\deg\,[a_1|\cdots|a_m] = (-m, \sum_i \operatorname{adeg} a_i).$$

The second component is the Adams degree of $[a_1|\cdots|a_m]$.

The bar construction on the left A-module A, denoted by $B(A,A)$, is constructed as follows. As a complex $B(A,A) = BA \otimes A$ with $(-i)$th term equal to $\mathfrak{m}^{\otimes i} \otimes A$. We use

$$[a_1|\cdots|a_m]x$$

to denote an element in $\mathfrak{m}^{\otimes i}\otimes A$ where $x \in A$ and $a_i \in \mathfrak{m}$. The degree of $[a_1|\cdots|a_m]x$ is $-m$. The differential on $B(A,A)$ is defined by

$$d(x) = 0 \quad (m = 0 \text{ case}),$$

and

$$d([a_1|\cdots|a_m]x) = \sum_{i=2}^{m}(-1)^{i-1}[a_1|\cdots|a_{i-1}a_i|\cdots|a_m]x + (-1)^m[a_1|\cdots|a_{m-1}]a_m x.$$

Then $B(A,A)$ is a complex of free right A-modules. Furthermore, the augmentations of BA and A make it into a free resolution of k_A,

$$B(A,A) \to k_A \to 0 \tag{4.1.1}$$

(see [FHT, 19.2] and [Ad, Chap.2]).

We now assume that with respect to the Adams grading, A is connected graded and finite-dimensional in each degree. Then $B(A,A)$ is bigraded with Adams grading on the second component, and the differential of $B(A,A)$ preserves the Adams grading. Let $B^{\#}A$ be the graded k-linear dual of the coalgebra BA. Since BA is locally finite, $B^{\#}A$ is a locally finite bigraded algebra. With respect to the Adams grading, $B^{\#}A$ is *negatively* connected graded. The DG algebra $\operatorname{End}_A(B(A,A)_A)$ is bigraded too, but not Adams connected. Since $B(A,A)$ is a left differential

graded comodule over BA, it has a left differential graded module structure over $B^{\#}A$, which is compatible with the right A-module structure. By an idea similar to [FHT, Ex. 4, p. 272] (also see [LP2]) one can show that the natural map $B^{\#}A \to \mathrm{End}_A(B(A,A)_A)$ is a quasi-isomorphism of DG algebras. Furthermore, the following lemma shows that, up to A_∞-isomorphism, $\mathrm{End}_A(P_A)$ is independent of the choice of resolution P_A, and hence the same goes for $\mathrm{Ext}_A(k_A, k_A)$.

Lemma 4.2. *Let A be a connected graded algebra which is finite-dimensional in each degree, and let P_A and Q_A be two free resolutions of k_A.*

(a) $\mathrm{End}_A(P_A)$ *is quasi-isomorphic to* $\mathrm{End}_A(Q_A)$ *as A_∞-algebras.*
(b) $\mathrm{End}_A(P_A)$ *is quasi-isomorphic to $B^{\#}A$ as A_∞-algebras.*

Proof. (a) We may assume that Q_A is a minimal free resolution of k_A. Then $P_A = Q_A \oplus I_A$ where I_A is another complex of free modules such that $HI_A = 0$. In this case $D := \mathrm{End}_A(Q_A)$ is a sub-DG algebra of $E := \mathrm{End}_A(P_A)$ such that $D = eEe$ where e is the projection onto Q_A. Let $C = (1-e)E + E(1-e)$. Then

$$C = \mathrm{Hom}_A(I_A, Q_A) + \mathrm{Hom}_A(Q_A, I_A) + \mathrm{Hom}_A(I_A, I_A),$$

and $HC = 0$. By Lemma 2.6, D and E are quasi-isomorphic.

(b) Since $B(A,A)$ is a free resolution of k_A, then part (a) says that $\mathrm{End}_A(P_A)$ is quasi-isomorphic to $\mathrm{End}_A(B(A,A)_A)$. The assertion follows from the fact that $\mathrm{End}_A(B(A,A))$ is quasi-isomorphic to $B^{\#}A$ [FHT, Ex. 4, p. 272]. □

The classical Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$ is the cohomology ring of $\mathrm{End}_A(P_A)$, where P_A is any free resolution of k_A. By the lemma, we may use $B^{\#}A$ instead. Since $B^{\#}A$ is a DG algebra, then by Theorem 2.2 and Proposition 2.3, $\mathrm{Ext}^*_A(k_A, k_A)$ has an A_∞-structure, which is called an *A_∞-Ext-algebra* of A. By abuse of notation we use $\mathrm{Ext}^*_A(k_A, k_A)$ to denote an A_∞-Ext-algebra. In the next section, we use $B^{\#}A$ and Merkulov's construction to produce an A_∞-algebra structure on $\mathrm{Ext}^*_A(k_A, k_A)$, and we use that to prove our main theorem.

5. A_∞-structure on Ext-algebras

In this section we consider the multiplications on an A_∞-Ext-algebra of a connected graded algebra, and finally give proofs of Theorem A and Proposition 1.2. Consider a connected graded algebra

$$A = k \oplus A_1 \oplus A_2 \oplus \cdots,$$

which is viewed as an A_∞-algebra concentrated in degree 0, with the grading on A being the Adams grading. Let $Q \subset A$ be a minimal graded vector space which generates A. Then $Q \cong \mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} := A_{\geq 1}$ is the unique maximal graded ideal of A. Following Milnor and Moore [MM, 3.7], we call the elements of $\mathfrak{m}/\mathfrak{m}^2$ the *indecomposables* of A, and by abuse of notation, we also call the elements of Q indecomposables. Let $R \subset T\langle Q\rangle$ be a minimal graded vector space which generates the relations of A (R is not unique). Then $A \cong T\langle Q\rangle/(R)$ where (R) is the ideal generated by R, and the start of a minimal graded free resolution of the trivial right A-module k_A is

$$\cdots \to R \otimes A \to Q \otimes A \to A \to k \to 0. \tag{5.0.1}$$

Lemma 5.1. *Let A be a connected graded algebra. Then there are natural isomorphisms of graded vector spaces*

$$\mathrm{Ext}^1_A(k_A, k_A) \cong Q^\# = \bigoplus Q^\#_s \quad \textit{and} \quad \mathrm{Ext}^2_A(k_A, k_A) \cong R^\# = \bigoplus R^\#_s.$$

Proof. This follows from the minimal free resolution (5.0.1). □

In the rest of the section, we assume that A is Adams locally finite: each A_i is finite-dimensional. Let E be the A_∞-Ext-algebra $\mathrm{Ext}^*_A(k_A, k_A)$. We would like to describe the A_∞-structure on E by using Merkulov's construction.

We first fix some notation. For each Adams degree s, we choose a vector space splitting $A_s = Q_s \oplus D_s$; the elements in Q_s are indecomposable, while those in D_s are "decomposable" in terms of the indecomposables of degree less than s. More precisely, we define Q_s and D_s inductively: we start by setting $Q_1 = A_1$ and $D_1 = 0$. Now assume that Q_i and D_i have been defined for $i < s$; then we set D_s to be the image in A_s of the multiplication map

$$\mu_s : \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \to A_s.$$

We choose Q_s to be a vector space complement of D_s.

Now for each $s \ge 2$, the multiplication map

$$\mu_s : \bigoplus_{1\le i\le s} Q_i \otimes A_{s-i} \to A_s$$

is onto. Define the k-linear map $\xi_s : A_s \to \bigoplus_{1\le i\le s} Q_i \otimes A_{s-i}$ so that the composition

$$A_s \xrightarrow{\xi_s} \bigoplus_{1\le i\le s} Q_i \otimes A_{s-i} \xrightarrow{-\mu_s} A_s \tag{5.1.1}$$

is the identity map of A_s. Further, we choose ξ_s so that with respect to the direct sum decomposition $A_s = Q_s \oplus D_s$, we have

$$\mathrm{im}(\xi_s|_{Q_s}) = Q_s \otimes A_0, \quad \mathrm{im}(\xi_s|_{D_s}) \subset \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}. \tag{5.1.2}$$

(The second of these holds for any choice of ξ_s; the first need not.) Define $\xi_1 = \theta_1 = \mathrm{id}_{A_1}$, and inductively set $\theta_s = \sum_{i+j=s}(\mathrm{id}_{Q_i} \otimes \theta_j) \circ \xi_s$; that is, θ_s is the composition

$$A_s \xrightarrow{\xi_s} \bigoplus_{i+j=s} Q_i \otimes A_j \xrightarrow{\sum \mathrm{id}_{Q_i}\otimes\xi_j} \bigoplus_{i+k+l=s} Q_i \otimes Q_k \otimes A_l \xrightarrow{\sum \mathrm{id}_{Q_i}\otimes\mathrm{id}_{Q_k}\otimes\xi_l} \cdots$$
$$\longrightarrow \bigoplus_{n\ge1} \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$

Here, the subscripts on the Q's are positive, while those on the A's are non-negative.

Let $R = \bigoplus_{s\ge2} R_s \subset T\langle Q\rangle$ be a minimal graded vector space of the relations of A. Note that with respect to tensor length, the elements of R need not be homogeneous, but they are homogeneous with respect to the Adams grading – the grading induced by that on Q.

Let $T\langle Q\rangle_s$ denote the part of $T\langle Q\rangle$ in Adams degree s; thus

$$T\langle Q\rangle_s = \bigoplus_{n\ge1} \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$

We write μ for the map $\mu : T\langle Q\rangle \to A \cong T\langle Q\rangle/(R)$.

Lemma 5.2. *For each s, R_s may be chosen so that*

$$R_s \subset \bigoplus_{1\le i\le s-1} Q_i \otimes \theta_{s-i}(A_{s-i}) \subset T\langle Q\rangle_s.$$

Hence R_s may also be viewed as a subspace of $\bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}$, via the composite

$$R_s \hookrightarrow \bigoplus_{1\le i\le s-1} Q_i \otimes \theta_{s-i}(A_{s-i}) \xrightarrow{\sum id_{Q_i}\otimes\mu} \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}.$$

Proof. Let $(R)_{s-i}$ be the degree $s-i$ part of the ideal (R). Then it is generated by the relations of degree at most $s-i$, and we have a decomposition

$$T\langle Q\rangle_{s-i} = \theta_{s-i}(A_{s-i}) \oplus \ker\mu = \theta_{s-i}(A_{s-i}) \oplus (R)_{s-i},$$

where $\mu : T\langle Q\rangle_{s-i} \to A_{s-i}$ is multiplication. Hence we have

$$T\langle Q\rangle_s = Q_s \oplus \bigoplus\nolimits_{1\le i\le s-1}\big[\big(Q_i \otimes \theta_{s-i}(A_{s-i})\big) \oplus \big(Q_i \otimes (R)_{s-i}\big)\big].$$

Any relation $r \in R_s$ has no summands in Q_s, and hence is a sum of $r' \in \bigoplus Q_i \otimes \theta_{s-i}(A_{s-i})$ and $r'' \in \bigoplus Q_i \otimes (R)_{s-i}$. Modulo the relations of degree less than n, we may assume $r'' = 0$. Hence the first part of the lemma is proved.

The map $\theta_{s-i} : A_{s-i} \to T\langle Q\rangle_{s-i}$ is an inclusion, and up to a sign its left inverse is the multiplication map $\mu : T\langle Q\rangle_{s-i} \to A_{s-i}$. Once R_s has been chosen to be a subspace of $\bigoplus Q_i \otimes \theta_{s-i}(A_{s-i})$, composing with μ takes it injectively to $\bigoplus Q_i \otimes A_{s-i}$. □

The minimal resolution (5.0.1) is a direct summand of any other resolution, and in particular it is a summand of the bar resolution (4.1.1)

$$\cdots \to \mathfrak{m}^{\otimes 2} \otimes A \to \mathfrak{m} \otimes A \to A \to k \to 0.$$

We have made several choices up to this point: choosing the splittings $A_s = Q_s \oplus D_s$, and now choosing R_s as in the lemma, so that $R_s \subset \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \subset \mathfrak{m}\otimes\mathfrak{m}$. These choices give a choice for this splitting of resolutions, at least in low degrees.

Since A is concentrated in degree 0, the grading on the bar construction $BA = T(S\mathfrak{m})$ is by the negative of the wordlength, namely, $(BA)^{-i} = \mathfrak{m}^{\otimes i}$. The differential $d = (d^i)$ on the bar construction is induced by the multiplication $\mathfrak{m}\otimes\mathfrak{m} \to \mathfrak{m}$ in A. For example,

$$d^{-1}([a_1]) = 0 \quad \text{and} \quad d^{-2}([a_1|a_2]) = (-1)^{-1}[a_1a_2]$$

for all $a_1, a_2 \in \mathfrak{m}$. There is a natural decomposition of $\mathfrak{m}$ with respect to the Adams grading,

$$\mathfrak{m} = A_1 \oplus A_2 \oplus A_3 \oplus \cdots,$$

which gives rise to a decomposition of $\mathfrak{m}\otimes\mathfrak{m}$ with respect to the Adams grading:

$$\mathfrak{m}\otimes\mathfrak{m} = (A_1\otimes A_1) \oplus (A_1\otimes A_2 \oplus A_2 \otimes A_1) \oplus \cdots.$$

As mentioned above, we are viewing R_s as being a subspace of $\bigoplus Q_i \otimes A_{s-i}$.

Lemma 5.3. *Let W_s be the Adams degree s part of $\mathfrak{m}\otimes\mathfrak{m}$; that is, let*

$$W_s = \bigoplus\nolimits_{1\le i\le s-1} A_i \otimes A_{s-i}.$$

Then there are decompositions of vector spaces

$$W_s = \operatorname{im}(d_s^{-3}) \oplus R_s \oplus \xi_s(D_s),$$
$$\ker(d_s^{-2}) = \operatorname{im}(d_s^{-3}) \oplus R_s,$$

where R_s and $\xi_s(D_s)$ are subspaces of

$$\bigoplus\nolimits_{1\le i\le s-1} Q_i \otimes A_{s-i} \subset \bigoplus\nolimits_{1\le i\le s-1} A_i \otimes A_{s-i} = W_s.$$

Proof. It is clear that the injection

$$D_s \xrightarrow{\xi_s} \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \longrightarrow W_s$$

defines a projection from W_s to D_s. Since $d_s^{-2} : W_s \to D_s$ is a surjection, we have a decomposition

$$W_s = \ker(d_s^{-2}) \oplus \xi_s(D_s).$$

Since $R^\# \cong \mathrm{Ext}^2_A(k_A, k_A) = H^2(B^\# A)$ by Lemma 5.1, there is a decomposition $\ker(d_s^{-2}) = \mathrm{im}(d_s^{-3}) \oplus R_s$. Hence the assertion follows. □

Since A is Adams locally finite, $(\mathfrak{m}^{\otimes n})^\# \cong (\mathfrak{m}^\#)^{\otimes n}$ for all n. Let $\Omega A^\#$ be the cobar construction on the DG coalgebra $A^\#$. Via the isomorphisms

$$B^\# A = (T(S\mathfrak{m}))^\# \cong T((S\mathfrak{m})^\#) \cong T(S^{-1}\mathfrak{m}^\#) = \Omega A^\#,$$

we identify $B^\# A = (T(S\mathfrak{m}))^\#$ with $\Omega A^\# = T(S^{-1}\mathfrak{m}^\#)$. The differential ∂ on $B^\# A$ is defined by

$$\partial(f) = -(-1)^{\deg f} f \circ d$$

for all $f \in T(S^{-1}\mathfrak{m}^\#)$.

We now study the first two nonzero differential maps of $\Omega A^\#$,

$$\partial^1 : \mathfrak{m}^\# \to (\mathfrak{m}^\#)^{\otimes 2} \qquad \text{and} \qquad \partial^2 : (\mathfrak{m}^\#)^{\otimes 2} \to (\mathfrak{m}^\#)^{\otimes 3}.$$

For all s and n, let

$$T^n = (\mathfrak{m}^\#)^{\otimes n}$$

and

$$T^n_{-s} = \bigoplus_{i_1+\cdots+i_n=s} A^\#_{i_1} \otimes \cdots \otimes A^\#_{i_n}.$$

Note that terms like $A^\#_i$ should be read as $(A_i)^\#$; hence (since we are working with $\mathfrak{m}$), all subscripts here and in what follows are positive. Fix Adams degree $-s$, and consider

$$\partial^1_{-s} : A^\#_s \to \bigoplus_{i+j=s} A^\#_i \otimes A^\#_j,$$
$$\partial^2_{-s} : \bigoplus_{i+j=s} A^\#_i \otimes A^\#_j \to \bigoplus_{i_1+i_2+i_3=s} A^\#_{i_1} \otimes A^\#_{i_2} \otimes A^\#_{i_3}.$$

The decomposition (2.1.1) for T^1_{-s} is

$$B^1_{-s} = 0, \quad H^1_{-s} = Q^\#_s, \quad L^1_{-s} = D^\#_s.$$

for all $s \ge 1$. The decomposition (2.1.1) for T^2_{-s} is given in the following lemma.

Lemma 5.4. *Fix $s \ge 2$. With notation as above, we have the following.*

(a) *Define the duals of subspaces by using the decompositions given in Lemma 5.3. Then $\mathrm{im}\,\partial^1_{-s} = (\xi_s(A_s))^\#$ and $\ker \partial^2_{-s} = (\xi_s(A_s))^\# \oplus R^\#_s$.*

(b) *The decomposition (2.1.1) for T^2_{-s} can be chosen to be*

$$T^2_{-s} = B^2_{-s} \oplus H^2_{-s} \oplus L^2_{-s} = (\xi_s(A_s))^\# \oplus R^\#_s \oplus (\mathrm{im}\, d_s^{-3})^\#.$$

The projections onto $R^\#_s$ and $(\xi_s(A_s))^\#$ kill $\bigoplus_{2\le i\le s-1} D^\#_i \otimes A^\#_{s-i}$.

(c) *Let G be the homotopy defined in Merkulov's construction for the DG algebra $T(S^{-1}\mathfrak{m}^\#)$. Then we may choose G^2_{-s} to be equal to $-(\xi_s)^\#$, restricted to T^2_{-s}.*

Proof. (a) This follows from Lemma 5.3 and a linear algebra argument.

(b) This follows from Lemma 5.3, part (a), and the fact that R_s and $\xi_s(A_s)$ are subspaces of $\bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}$.

(c) Let ξ_s also denote the map $D_s \to \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \to W_s$. Since $d_s^{-2} = -\mu_s$, the composite $d_s^{-2} \circ \xi_s : D_s \to W_s \to D_s$ is the identity map (see (5.1.1)). Since $(d_s^{-2} \circ \xi_s)^\# = (\xi_s)^\# \circ (d_s^{-2})^\#$,

$$(\xi_s)^\# \circ (d_s^{-2})^\# : \ D_s^\# \to T^2_{-s} \to D_s^\#$$

is the identity map of $D_s^\#$, which is the summand L^1_{-s} of T^1. Since $(d_s^{-2})^\# = -\partial^1_{-s}$, we may choose the homotopy G to be $-(\xi_s)^\#$ when restricted to T^2_{-s}. □

Now we start to construct the higher A_∞-multiplication maps on $\mathrm{Ext}^*_A(k_A, k_A)$, using Merkulov's construction from Section 2. Lemma 5.4 tells us what the homotopy G is. The maps $\lambda_n : (T(S^{-1}\mathfrak{m}^\#))^{\otimes n} \to T(S^{-1}\mathfrak{m}^\#)$ are defined as in Section 2; in particular, recall that we formally set $G\lambda_1 = -\mathrm{id}_T$, and λ_2 is the multiplication of $T(S^{-1}\mathfrak{m}^\#)$.

Recall that $T(S^{-1}\mathfrak{m}^\#)$ is a free (or tensor) DG algebra generated by $S^{-1}\mathfrak{m}^\#$. To distinguish among the various tensor products occurring here, we use $\bigotimes$ when tensoring factors of $T(S^{-1}\mathfrak{m}^\#)$ together; in particular, we write λ_2 as

$$\lambda_2 : (S^{-1}\mathfrak{m}^\#)^{\otimes n} \bigotimes (S^{-1}\mathfrak{m}^\#)^{\otimes m} \to (S^{-1}\mathfrak{m}^\#)^{\otimes (n+m)}.$$

We also use tensors $\otimes$ rather than bars $|$ for terms in the bar construction.

For $a_1 \otimes \cdots \otimes a_n \in (S^{-1}\mathfrak{m}^\#)^{\otimes n}$ and $b_1 \otimes \cdots \otimes b_m \in (S^{-1}\mathfrak{m}^\#)^{\otimes m}$ we have

$$\lambda_2((a_1 \otimes \cdots \otimes a_n) \bigotimes (b_1 \otimes \cdots \otimes b_m)) = a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_m. \tag{5.4.1}$$

By the above formula, we see that λ_2 changes $\bigotimes$ to $\otimes$, so it is like the identity map.

Lemma 5.5. *Let $E^1 = \mathrm{Ext}^1_A(k_A, k_A) = Q^\#$. Fix $n \ge 2$ and $s \ge 2$.*

(a) *When restricted to $(E^1)^{\otimes n}$ in Adams degree $-s$, the map λ_n has image in*

$$T^2_{-s} = \bigoplus_{q+r=s} A_q^\# \otimes A_r^\#.$$

Hence the image of $G\lambda_n$ is in $D_s^\#$.

(b) *When restricted to $(E^1)^{\otimes n}$ in Adams degree $-s$, the map $-G\lambda_n$ is the k-linear dual of the composite*

$$A_s \xrightarrow{\theta_s} \bigoplus_{m\ge 1} \bigoplus_{i_1+\cdots+i_m=s} Q_{i_1} \otimes \cdots \otimes Q_{i_m} \to \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$

(c) *When restricted to $(E^1)^{\otimes n}$ in Adams degree $-s$, the map $m_n = Pr_H \lambda_n$ is the k-linear dual of the canonical map*

$$R_s \longrightarrow \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \xrightarrow{\sum id\otimes\theta_{s-i}} \bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$

Proof. We use induction on n.

(a) By definition,

$$\lambda_n = \lambda_2 \sum_{\substack{i+j=n \\ i,j>0}} (-1)^{i+1} G\lambda_i \bigotimes G\lambda_j.$$

For $n = 2$, the claim follows from (5.4.1). Now assume $n > 2$ and consider λ_n applied to $b_1 \otimes \cdots \otimes b_n$, with $b_m \in Q^\#_{i_m}$ for each m. By Lemma 5.4(c), when restricted

to T^2_{-r} for any r, $-G$ is dual to the map $\xi_r : A_r \to \bigoplus Q_i \otimes A_{r-i} \subset \bigoplus A_i \otimes A_{r-i}$, so by induction, for each $m < n$,

$$G\lambda_m(b_1 \otimes \cdots \otimes b_m) \in G(\bigoplus_{i+j=i_1+\cdots+i_m} A_i^\# \otimes A_j^\#) \subset A^\#_{i_1+\cdots+i_m},$$

and similarly for $G\lambda_{n-m}(b_{m+1} \otimes \cdots \otimes b_n)$. Thus the first statement follows. The second statement follows from the assumption (5.1.2) about the map ξ_n.

(b) When $n = 2$, $\theta_2 = \xi_2$, and the claim follows from Lemma 5.4(c). Now we assume $n > 2$. When restricted to $(E^1)^{\otimes n}$ in Adams degree $-s$, part (a) says that if $i > 1$, then the image of $G\lambda_i \bigotimes G\lambda_j$ is in $\bigoplus_{q+r=s} D_q^\# \bigotimes D_r^\# \subset \bigoplus_{q+r=s} D_q^\# \bigotimes A_r^\#$. By Lemma 5.4(b,c),

$$G\lambda_2(D_q^\# \bigotimes A_r^\#) = G(D_q^\# \otimes A_r^\#) = -(\xi_n)^\#(D_q^\# \otimes A_r^\#) = 0.$$

Therefore, when restricted to $((E^1)^{\otimes n})_{-s}$, we have

$$G\lambda_n = G\lambda_2[(-1)^2(-\mathrm{id}) \bigotimes G\lambda_{n-1}] = -G\lambda_2(\mathrm{id} \bigotimes G\lambda_{n-1}).$$

By induction on n, in any Adams degree $-r$, $G\lambda_{n-1}$ is $-(\theta_r)^\#$ composed with projection to a summand, and by Lemma 5.4(c) we see that in Adams degree $-q$, we have $G = -(\xi_q)^\#$. Hence when applied to $b_1 \otimes b_2 \otimes \cdots \otimes b_n$ with b_1 in Adams degree $-i$ and $b_2 \otimes \cdots \otimes b_n$ in Adams degree $i - s$, we have

$$G\lambda_n = (\xi_s)^\# \lambda_2(\mathrm{id}_{E^1_{-i}} \bigotimes -(\theta_{s-i})^\#) = -((\mathrm{id}_{Q_i} \otimes \theta_{s-i}) \circ \xi_s)^\#,$$

and thus $G\lambda_n$ is exactly $-\theta_s^\#$ followed by projection onto the tensor length n summand

$$\bigoplus_{i_1+\cdots+i_n=s} Q_{i_1} \otimes \cdots \otimes Q_{i_n}.$$

(c) Since we assume that R_s is a subspace of $\bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}$, the dual of the inclusion

$$R_s \to \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}$$

is Pr_H restricted to $\left(\bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}\right)^\#$. Hence the dual of

$$R_s \to \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i} \to (Q \otimes Q^{\otimes(n-1)})_s$$

is equal to $Pr_H \circ (\sum \mathrm{id} \otimes \theta_{s-i})^\#$.

By Lemma 5.4(b), Pr_H is zero when applied to $D_q^\# \otimes A_r^\#$ for all q. By (b),

$$Pr_H\lambda_n = Pr_H\lambda_2\left(-\sum_{q+r=s} \mathrm{id}_{Q_q^\#} \bigotimes G\lambda_r\right) = Pr_H\left(\sum_{q+r=s} \mathrm{id}_{E^1_q} \otimes (\theta_r)^\#\right),$$

which is the desired map. □

Proof of Theorem A. First of all by Lemma 5.2, we may assume that

$$R_s \subset \bigoplus_{1\le i\le s-1} Q_i \otimes A_{s-i}.$$

Then we appeal to Lemmas 5.3, 5.4 and 5.5. The canonical map in Lemma 5.5(c) is just the inclusion, and so the assertion holds. □

Proof of Corollary B. Note that under the assumptions in the corollary, the vector space of indecomposables is canonically isomorphic to A_1. Thus various parts of Theorem A simplify; for example, E^1 is isomorphic to $A_1^\#$, and hence is concentrated in Adams degree -1. □

The following corollary is immediate.

Corollary 5.6. *Let A and E be as in Theorem A.*

(a) *The algebra A is determined by the maps m_n restricted to $(E^1)^{\otimes n}$ for all n.*
(b) *The A_∞-structure of E is determined up to quasi-isomorphism by the maps m_n restricted to $(E^1)^{\otimes n}$ for all n.*

Proof. (a) By Theorem A, the map $R \to T\langle Q\rangle$ can be recovered from m_n restricted to $(E^1)^{\otimes n}$. Hence the structure of A is determined.

(b) After A is recovered, the A_∞-structure of E is determined by A. Therefore the structure of E is determined by the restriction of m_n on $(E^1)^{\otimes n}$, up to quasi-isomorphism. □

Proof of Proposition 1.2. By the construction given above, it is clear that if the grading group for the Adams grading is $\mathbb{Z} \oplus G$ for some abelian group G, then all of the maps including m_n preserve the G-grading. The assertion follows. □

6. Examples

Several examples of A_∞-algebras E are given in [LP1, Exs. 3.5, 3.7, 13.4 and 13.5]. We conclude this paper with a few more examples. Theorem A says that there is a choice of A_∞-algebra structures on $E = \mathrm{Ext}^*_A(k_A, k_A)$ in which the higher multiplications reflect the relations in A. Sometimes, though, the grading forces the A_∞-algebra structure to be unique, at least when restricted to tensor powers of E^1. Of course the ordinary multiplication m_2 is the same for any choice. So consider m_3, for example. Suppose that there are two A_∞-algebra structures $(E, m_2, m_3, \dots)$ and $(E, m_2', m_3', \dots)$ with $m_2 = m_2'$ and with an A_∞-isomorphism

$$f : (E, m_2, m_3, \dots) \to (E, m_2', m_3', \dots)$$

with $f_1 = \mathrm{id}$. The morphism identity **MI(3)** applied to $\alpha \otimes \beta \otimes \gamma \in (E^1)^{\otimes 3}$ says

$$\begin{aligned} m_3(\alpha \otimes \beta \otimes \gamma) - f_2(\alpha \otimes \beta\gamma) + f_2(\alpha\beta \otimes \gamma) \\ = m_3'(\alpha \otimes \beta \otimes \gamma) - f_2(\alpha\beta)\gamma - \alpha f_2(\beta\gamma). \end{aligned} \tag{6.0.1}$$

Depending on the Adams grading, degree reasons may imply that the various applications of f_2 are all zero, in which case m_3 would agree with m_3' on $(E^1)^{\otimes 3}$. For example, if E^1 and E^2 are each concentrated in a single nonzero Adams degree, then any A_∞-isomorphism f as above will have $f_i = 0$ on tensor powers of E^1, and also on terms of the form $(E^1)^{\otimes r} \otimes E^2 \otimes (E^1)^{\otimes t}$. These are precisely the sort of terms which arise in **MI(n)** applied to $(E^1)^{\otimes n}$, and so one can conclude that the A_∞-algebra structure is unique when restricted to tensor powers of E^1. This is the situation in the first and third examples below, and the second example can be dealt with similarly. (We deal with uniqueness in the fourth example separately. The fifth example does not invoke Theorem A, so the issue does not arise.)

Example 6.1. Fix a field k and consider the free algebra $B = k\langle x_1, x_2\rangle$ on two generators, each in Adams degree 1. Fix an integer $q \geq 2$, let $f(x_1, x_2)$ be an element in Adams degree q, and let $A = B/(f)$. Then the minimal resolution (5.0.1) for k_A has the form

$$\cdots \to Ar \xrightarrow{i} Ae_1 \oplus Ae_2 \to A \to k_A \to 0,$$

where the generator e_i corresponds to x_i, and r corresponds to $f(x_1, x_2)$. Order the monomials in B left-lexicographically, setting $x_1 < x_2$, and assume that with

respect to this ordering, $f(x_1, x_2)$ has leading term $x_2^i x_1^{q-i}$ with $0 < i < q$. Then one can show that the map $i : Ar \to Ae_1 \oplus Ae_2$ is injective, so

$$\mathrm{Ext}_A^s(k_A, k_A) = \begin{cases} k & s = 0, \\ k(-1) \oplus k(-1) & s = 1, \\ k(-q) & s = 2, \\ 0 & \text{else.} \end{cases}$$

Ext^1 is dual to A_1, and we choose (y_1, y_2) to be the dual basis to (x_1, x_2). We write z for the generator of Ext^2 dual to f. For degree reasons, the A_∞-algebra structure on Ext has the property that $m_n = 0$ unless $n = q$. Also for degree reasons, the A_∞-structure is unique: any A_∞-isomorphism

$$f : (E, m_2, m_3, m_4, \dots) \to (E, m_2', m_3', m_4', \dots)$$

as above must have $f_n = 0$ for $n > 1$. By Theorem A, the map m_q is "dual to the relations":

$$m_q(y_{i_1} \otimes \cdots \otimes y_{i_q}) = \alpha z \quad \text{if } \alpha x_{i_1} \cdots x_{i_q} \text{ is a summand in } f(x_1, x_2).$$

So for example, if $q > 2$, then as an associative algebra, $\mathrm{Ext}_A^*(k_A, k_A)$ has trivial multiplication no matter what f is, so one cannot recover A from the ordinary algebra structure. One can recover A from the A_∞-algebra structure, though.

Example 6.2. Let $A = k[x_2, x_3]/(x_3^2 - x_2^3)$, graded by giving each x_i Adams degree i. The graded vector space Q has two nonzero graded pieces: Q_i is spanned by x_i when $i = 2, 3$. Let (b_2, b_3) be the graded basis for $Q^\#$ which is dual to (x_2, x_3). The space R of relations has two graded pieces also: there is the degree 5 relation $r_5 = x_2x_3 - x_3x_2$, and the degree 6 relation $r_6 = x_3^2 - x_2^3$. Let (s_5, s_6) be the graded basis for $R^\#$ which is dual to the basis (r_5, r_6). Thus in low dimensions, the Ext algebra is given by

$$\mathrm{Ext}_A^n(k_A, k_A) = \begin{cases} k & n = 0, \\ k(-2) \oplus k(-3) & n = 1, \\ k(-5) \oplus k(-6) & n = 2. \end{cases}$$

Indeed, by viewing A as a subalgebra of $k[y]$ (with $A \hookrightarrow k[y]$ defined by $x_i \mapsto y^i$), one can construct a minimal resolution for k_A to find that if $n > 0$,

$$\mathrm{Ext}_A^n(k_A, k_A) = k(-3n + 1) \oplus k(-3n),$$

with vector space basis $(b_2 b_3^{n-1}, b_3^n)$. Theorem A gives us the following formulas in Ext:

$$\begin{aligned} m_2(b_2 \otimes b_3) &= s_5, \\ m_2(b_3 \otimes b_2) &= -s_5, \\ m_2(b_3 \otimes b_3) &= s_6, \\ m_3(b_2 \otimes b_2 \otimes b_2) &= -s_6. \end{aligned}$$

All other instances of m_2 and m_3 on classes from E^1 are zero, for degree reasons.

Example 6.3. Fix $p > 2$ and let $A = k[x]/(x^p)$ with x in Adams degree $2d$ (so that A is graded commutative). Its Ext algebra is

$$\mathrm{Ext}_A^*(k_A, k_A) \cong \Lambda(y_1) \otimes k[y_2],$$

with y_i in Ext^i, with y_1 in Adams degree $-2d$ and y_2 in Adams degree $-2dp$. Then Theorem A tells us that we may choose y_1 and y_2 so that $m_p(y_1 \otimes \cdots \otimes y_1) = y_2$. It is a standard Massey product computation that the p-fold Massey product $\langle y_1, \dots, y_1 \rangle$ equals a generator of Ext^2 (with no indeterminacy, for degree reasons), and Theorem 3.1 tells us that with our choice of y_1 and y_2, we have $\langle y_1, \dots, y_1 \rangle = \{(-1)^{(p+1)/2} y_2\}$.

Example 6.4. Let k be a field of characteristic 2, and define the k-algebra A by

$$A = k\langle x_1, x_2 \rangle / (x_1^2,\, x_1 x_2 x_1 + x_2^2),$$

graded by putting x_i in Adams degree i. This is the sub-Hopf algebra $A(1)$ of the mod 2 Steenrod algebra, and its cohomology can be computed using spectral sequences – see Wilkerson [Wi, 2.4] or Ravenel [Ra, 3.1.25], for instance. In low degrees, it has

$$\mathrm{Ext}_A^n(k_A, k_A) = \begin{cases} k & n = 0, \\ k(-1) \oplus k(-2), & n = 1, \\ k(-2) \oplus k(-4), & n = 2. \end{cases}$$

Following Wilkerson, we write (h_0, h_1) for the basis of Ext^1, with h_i in Adams degree -2^i. Then Theorem A tells us that $m_2(h_0 \otimes h_0) \neq 0$ and $m_2(h_1 \otimes h_1) \neq 0$, so (h_0^2, h_1^2) is a basis for Ext^2. Theorem A also gives the formula

$$m_3(h_0 \otimes h_1 \otimes h_0) = h_1^2.$$

This reflects the Massey product computation $\langle h_0, h_1, h_0 \rangle = \{h_1^2\}$. Note that in this case, degree reasons are not enough to force the A_∞-structure to be unique; for example, in an A_∞-isomorphism $f : (E, m_2, m_3, \dots) \to (E, m_2', m_3', \dots)$, there could be a nonzero f_2 from $E^1_{-1} \otimes E^1_{-1}$ to E^1_{-2}. However, $m_3(h_0 \otimes h_1 \otimes h_0)$ will be the same in any A_∞-structure: in equation (6.0.1) applied to $h_0 \otimes h_1 \otimes h_0$, each f_2 term involves $h_0 h_1$ or $h_1 h_0$, and these products are zero.

Example 6.5. We present a DG algebra whose homology has a Massey product $\langle \alpha, \beta, \gamma \rangle$ containing several elements, such that for any element ζ in that Massey product, one may choose an A_∞-algebra structure via Merkulov's construction so that $m_3(\alpha \otimes \beta \otimes \gamma) = \zeta$. Define a (singly-graded) DG algebra A by

$$A = k\langle a, b, c, x, y \rangle,$$

with all generators in degree 1, and with differential determined by

$$d(a) = d(b) = d(c) = 0, \quad d(x) = ab, \quad d(y) = bc.$$

Then a, b, and c represent cohomology classes, which we call α, β, and γ, respectively. Because ab and bc are coboundaries, the Massey product $\langle \alpha, \beta, \gamma \rangle$ is defined, and contains the cohomology class θ defined by

$$\theta = [xc + ay].$$

However, x may be replaced with $x + z$ for any cocycle z in degree 1, and similarly for y. Since $bc = 0$, replacing x with $x + b$ has no effect in homology, and similarly for y. Thus we have

$$\langle \alpha, \beta, \gamma \rangle = \{[(x + ra + sc)c + a(y + ta + uc)] \,:\, r, s, t, u \in k\},$$

where the square brackets denote the cohomology class. The term uac is redundant, so

$$\langle \alpha, \beta, \gamma \rangle = \{\theta + r\alpha\gamma + s\gamma^2 + t\alpha^2 \,:\, r, s, t \in k\}.$$

Note that since the space of boundaries in degree 2 is spanned by ab and bc, the cohomology classes listed here are distinct.

Now we claim that for any element ζ in this Massey product, there is a Merkulov model for HA with $m_3(\alpha \otimes \beta \otimes \gamma) = \zeta$. Let $\zeta = \theta + r\alpha\gamma + s\gamma^2 + t\alpha^2$. To perform the construction, we choose splittings as in (2.1.1) of A^1 and A^2:

$$\begin{aligned} A^1 &= B^1 \oplus H^1 \oplus L^1 \\ &= 0 \oplus \operatorname{Span}(a, b, c) \oplus \operatorname{Span}(x + ra + sc, y + ta), \\ A^2 &= B^2 \oplus H^2 \oplus L^2 \\ &= \operatorname{Span}(ab, bc) \oplus H^2 \oplus L^2. \end{aligned}$$

The map $G^2 : A^2 \to A^1$ is zero on $H^2 \oplus L^2$, and on B^2, it is the inverse of $\partial^1 : L^1 \to B^2$. Thus it satisfies

$$G(ab) = x + ra + sc, \quad G(bc) = y + ta.$$

By the computation of m_3 in the proof of Theorem 3.1, we have the following, where p is the composite of the projection map from A^2 to its summand H^2, followed by the canonical identification of H^2 with $H^2(A)$:

$$\begin{aligned} m_3(\alpha \otimes \beta\alpha \otimes \gamma) &= p\left((x + ra + sc)c + a(y + ta)\right) \\ &= p\left((xc + ay) + rac + sc^2 + ta^2\right) \\ &= \theta + r\alpha\gamma + s\gamma^2 + t\alpha^2, \end{aligned}$$

as desired.

Appendix A. More on Merkulov's construction

The main result of this appendix is Theorem A.4, in which we show that any A_∞-algebra E with $m_1 = 0$ arises from a Merkulov model for some DG algebra. As a consequence, we show that the result from our Massey product theorem, Theorem 3.1, is independent of the choice of A_∞-structure: see Corollary A.5 below. This material is relegated to an appendix because it requires some machinery which is not needed in the rest of the paper. In particular, it uses the bar construction for A_∞-algebras, and it uses the notion of an enveloping algebra for an A_∞-algebra.

We discussed the bar construction for DG algebras in Section 4. This construction may be generalized to apply to A_∞-algebras. See Keller [Ke3, Section 3.6] for more details and references; this is also presented in [LP1, Section 9]. We need the bar construction for augmented A_∞-algebras, which may be obtained from Keller's version using the equivalence of categories between A_∞-algebras and augmented A_∞-algebras, as in [Ke3, Section 3.5].

Definition A.1. Let $(A, m_1, m_2, m_3, \dots)$ be an augmented A_∞-algebra with augmentation ideal $\mathfrak{m}$. The (augmented A_∞) *bar construction* $B_\infty^{\mathrm{aug}} A$ on A is a coaugmented DG coalgebra, defined as follows: as a coaugmented graded coalgebra, it is the tensor coalgebra on $S\mathfrak{m}$:

$$B_\infty^{\mathrm{aug}} A = T(S\mathfrak{m}) = k \oplus (S\mathfrak{m}) \oplus (S\mathfrak{m})^{\otimes 2} \oplus (S\mathfrak{m})^{\otimes 3} \oplus \cdots .$$

As is standard, we use bars rather than tensors, and we also conceal the shift S, writing $[a_1|\cdots|a_m]$ for the element $Sa_1 \otimes \cdots \otimes Sa_m$, where $a_i \in \mathfrak{m}$ for each i. The

degree of this element is

$$\deg[a_1|\cdots|a_m] = \sum(-1+\deg a_i).$$

The differential b on $B_\infty^{\mathrm{aug}}A$ is defined as follows: its component $b_m : (S\mathfrak{m})^{\otimes m} \to T(S\mathfrak{m})$ is given by

$$\begin{aligned} &b_m([a_1|\cdots|a_m]) \\ &\quad = \sum_{j,n}(-1)^{w_{j,n}+n}[a_1|\cdots|a_j|m_n(a_{j+1}\otimes\cdots\otimes a_{j+n})|a_{j+n+1}|\cdots|a_m] \end{aligned} \tag{A.1.1}$$

where

$$w_{j,n} = \sum_{1\le s\le j}(-1+\deg a_s) + \sum_{1\le t<n}(n-t)(-1+\deg a_{j+t}).$$

The differential b encodes all of the higher multiplications of A into a single operation. Keller [Ke3, 3.6] notes that if A and A' are augmented A_∞-algebras, then there is a bijection between Hom sets

$$\mathsf{Alg}_\infty^{\mathrm{aug}}(A,A') \longleftrightarrow \mathsf{DGC}^{\mathrm{coaug}}(B_\infty^{\mathrm{aug}}A, B_\infty^{\mathrm{aug}}A'). \tag{A.1.2}$$

We also need some details about the cobar construction.

Definition A.2. Given a coaugmented DG coalgebra (C, d_C) with coaugmentation coideal $\overline{C} = \mathrm{cok}(k \to C)$, the *cobar construction* on C is the augmented DG algebra ΩC defined as follows: as an augmented algebra, it is the tensor algebra on $S^{-1}\overline{C}$, and its differential is the sum $d = d_0 + d_1$ of the differentials

$$d_0\,(x_1\otimes\cdots\otimes x_m) = -\sum_{i=1}^{m}(-1)^{n_i}x_1\otimes\cdots\otimes d_C(x_i)\otimes\cdots\otimes x_m,$$

and

$$d_1\,(x_1\otimes\cdots\otimes x_m) = \sum_{i=1}^{m}\sum_{j=1}^{k_i}(-1)^{n_i+\deg a_{ij}}x_1\otimes\cdots\otimes x_{i-1}\otimes a_{ij}\otimes b_{ij}\otimes x_{i+1}\otimes\cdots\otimes x_m$$

where $x_i \in \overline{C}$, $n_i = \sum_{j<i}(1+\deg x_j)$ and $\overline{\Delta}x_i = \sum_{j=1}^{k_i} a_{ij}\otimes b_{ij}$. (As with the bar construction, we conceal the shift S^{-1}. Also, it is more usual to use bars, not tensors, for the cobar construction, but since we are about to combine this with the bar construction, we want different notation for the two.) Since ΩC is the tensor algebra on $S^{-1}\overline{C}$, the element $x_1\otimes\cdots\otimes x_m$ lies in degree $\sum(1+\deg x_i)$.

Since the bar construction $B_\infty^{\mathrm{aug}}A$ is a coaugmented DG coalgebra, one may apply the cobar construction to it.

Definition A.3. [LH, Section 2.3.4] Let A be an augmented A_∞-algebra. Its *enveloping algebra* is the augmented DG algebra $\Omega B_\infty^{\mathrm{aug}}A$.

At the level of DG algebras and coalgebras, the functors Ω and B are adjoint, so there is a natural transformation $\mathrm{id} \to B\Omega$. When applied to the DG coalgebra $B_\infty^{\mathrm{aug}}A$, this yields a DG coalgebra map

$$B_\infty^{\mathrm{aug}}A \to B\Omega(B_\infty^{\mathrm{aug}}A).$$

Also, if R is an augmented DG algebra, then one can view it as an augmented A_∞-algebra with trivial higher multiplications, and in this case, the two bar constructions agree: $BR = B_\infty^{\mathrm{aug}} R$. Therefore we can write this DG coalgebra map as

$$B_\infty^{\mathrm{aug}} A \to B_\infty^{\mathrm{aug}} \Omega(B_\infty^{\mathrm{aug}} A).$$

Then the bijection (A.1.2) gives an A_∞-algebra map

$$A \to \Omega B_\infty^{\mathrm{aug}} A.$$

One can show that this is an A_∞-isomorphism – see [LH, 1.3.3.6 and 2.3.4.3]. Thus every A_∞-algebra is A_∞-isomorphic to a DG algebra.

Here is the main theorem of this section. It says that Merkulov models (Definition 2.4) are ubiquitous among A_∞-algebra structures with $m_1 = 0$.

Theorem A.4. *Let A be an augmented A_∞-algebra with $m_1 = 0$. Then A may be obtained as a Merkulov model of its enveloping algebra $\Omega B_\infty^{\mathrm{aug}} A$.*

Proof. First, notation: as indicated above, we write $[a_1|\cdots|a_n]$ for a basic tensor in $B_\infty^{\mathrm{aug}} A = T(S\mathfrak{m})$, and we write

$$[a_{11}|\cdots|a_{1,n_1}] \otimes [a_{21}|\cdots|a_{2,n_2}] \otimes \cdots \otimes [a_{m,1}|\cdots|a_{m,n_m}]$$

for a basic tensor in $\Omega B_\infty^{\mathrm{aug}} A = T(S^{-1}\overline{B_\infty^{\mathrm{aug}} A})$. We say that the *tensor length* of such an element is m, and the *bar length* is $n_1 + \cdots + n_m$. The degree of such an element is

$$(\text{tensor length}) - (\text{bar length}) + \sum \deg a_{ij}.$$

Note that depending on context, $[a_1|\cdots|a_n]$ can mean an element of $(S\mathfrak{m})^{\otimes n} \subseteq B_\infty^{\mathrm{aug}} A$ or an element of $(B_\infty^{\mathrm{aug}} A)^{\otimes 1} \subseteq \Omega B_\infty^{\mathrm{aug}} A$. This distinction is most important in grading: for example, as an element of $B_\infty^{\mathrm{aug}} A$, $[a]$ has degree $-1 + \deg a$, while as an element of $\Omega B_\infty^{\mathrm{aug}} A$, $[a]$ has degree $1 - 1 + \deg a = \deg a$.

Now suppose we are given an augmented A_∞-algebra $(A, m_2^A, m_3^A, m_4^A, \dots)$ with $m_1^A = 0$. To perform Merkulov's construction on $\Omega B_\infty^{\mathrm{aug}} A$, we need to choose a splitting

$$\Omega B_\infty^{\mathrm{aug}} A = B \oplus H \oplus L,$$

as in (2.1.1); we want the resulting multiplication maps $m_n^H : H^{\otimes n} \to H$ to agree with m_n^A, after identifying H with A. We choose H to be as close to A as possible: we let H be the vector space spanned by the unit $1 \in \Omega B_\infty^{\mathrm{aug}} A$, together with the classes $[a] \in \Omega B_\infty^{\mathrm{aug}} A$. For the L summand, we only require that L contains the vector space

$$L' = \mathrm{Span}\{[a_1|\cdots|a_n] \,:\, n \geq 2\}.$$

We have no other conditions on L; any choice containing L' will suffice. Now, we need to verify that L may be chosen to contain L'; that is, we need to verify that $d|_{L'}$ is injective. The differential d is defined by $d = d_0 + d_1$, as in Definition A.2. On basis elements of L', the component d_1 is given by

$$d_1[a_1|\cdots|a_n] = \sum_{i=1}^{n-1} (-1)^{\sum_{j\leq i}(-1+\deg a_j)} [a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n], \tag{A.4.1}$$

and $d_0[a_1|\cdots|a_n]$ is equal to $-b_n[a_1|\cdots|a_n]$, as given in equation (A.1.1). In particular, d_1 increases tensor length by 1 and preserves bar length, while d_0 preserves tensor length and decreases bar length by at least 1. Since d_1 is injective on L', we conclude that d is injective on L'.

The Merkulov multiplication map m_n^H is given by $m_n^H = p\lambda_n$, where p is projection onto H, and λ_n is defined recursively in (2.1.2). Since we want to show that

$$m_n^H([a_1] \otimes \cdots \otimes [a_n]) = [m_n^A(a_1 \otimes \cdots \otimes a_n)],$$

we need to express

$$\lambda_n([a_1] \otimes \cdots \otimes [a_n])$$

as a sum $b + h + \ell$ where $b \in B$, $h \in H$, and $\ell \in L$; and further, we want h to equal $[m_n^A(a_1 \otimes \cdots \otimes a_n)]$.

Given $n \geq 2$ and $a_1, \ldots, a_n \in A$, let $v_n = 1 + \sum_{s<n}(n-s)\deg a_s$. We will prove the following formula inductively on n, for $n \geq 2$.

$$\lambda_n([a_1] \otimes \cdots \otimes [a_n]) = (-1)^{v_n} d_1[a_1|\cdots|a_n]. \tag{A.4.2}$$

We claim that this formula implies the following:

$$\begin{aligned} Gd\lambda_n([a_1] \otimes \cdots \otimes [a_n]) &= -(-1)^{v_n} d_0[a_1|\cdots|a_n] \\ &\quad - [m_n^A(a_1 \otimes \cdots \otimes a_n)], \end{aligned} \tag{A.4.3}$$

$$\begin{aligned} \lambda_n([a_1] \otimes \cdots \otimes [a_n]) &= (-1)^{v_n} d[a_1|\cdots|a_n] \\ &\quad + [m_n^A(a_1 \otimes \cdots \otimes a_n)] \\ &\quad + Gd\lambda_n([a_1] \otimes \cdots \otimes [a_n]), \end{aligned} \tag{A.4.4}$$

$$p\lambda_n([a_1] \otimes \cdots \otimes [a_n]) = [m_n^A(a_1 \otimes \cdots \otimes a_n)], \tag{A.4.5}$$

$$G\lambda_n([a_1] \otimes \cdots \otimes [a_n]) = (-1)^{v_n}[a_1|\cdots|a_n]. \tag{A.4.6}$$

Note that since the image of G is L, (A.4.4) is the desired expression for $\lambda_n(\cdots)$ as a sum $b+h+\ell$, and (A.4.5) is exactly what we are trying to prove. First we explain how to derive these consequences from (A.4.2), and then we verify that formula.

Since the differential d is defined by $d = d_0 + d_1$, equation (A.4.2) implies

$$\begin{aligned} \lambda_n([a_1] \otimes \cdots \otimes [a_n]) &= (-1)^{v_n} d_1[a_1|\cdots|a_n] \\ &= (-1)^{v_n} d[a_1|\cdots|a_n] - (-1)^{v_n} d_0[a_1|\cdots|a_n] \\ &= (-1)^{v_n} d[a_1|\cdots|a_n] \\ &\quad - \left((-1)^{v_n} d_0[a_1|\cdots|a_n] + [m_n^A(a_1 \otimes \cdots \otimes a_n)]\right) \\ &\quad + [m_n^A(a_1 \otimes \cdots \otimes a_n)]. \end{aligned}$$

Now apply d to both sides of this equation: $d^2 = 0$, and also the term $[m_n^A(a_1 \otimes \cdots \otimes a_n)]$ is a cycle, so we find that

$$d\lambda_n([a_1] \otimes \cdots \otimes [a_n]) = -d\left((-1)^{v_n} d_0[a_1|\cdots|a_n] + [m_n^A(a_1 \otimes \cdots \otimes a_n)]\right).$$

Now, every summand in $d_0[a_1|\cdots|a_n]$ has tensor length one, and the only summand with bar length less than 2 is $(-1)^{v_n+1}[m_n^A(a_1 \otimes \cdots \otimes a_n)]$; thus

$$(-1)^{v_n} d_0[a_1|\cdots|a_n] + [m_n^A(a_1 \otimes \cdots \otimes a_n)]$$

is in $L' \subseteq L$. Since G takes any boundary to the unique class in L which hits it, we see that

$$Gd(\lambda_n([a_1] \otimes \cdots \otimes [a_n])) = -(-1)^{v_n} d_0[a_1|\cdots|a_n] - [m_n^A(a_1 \otimes \cdots \otimes a_n)].$$

This verifies equations (A.4.3) and (A.4.4). The function p is projection onto the summand H, and the function G is computed by projection to B and then mapping to the preimage in L of the resulting boundary. Since equation (A.4.4) expresses

$\lambda_n([a_1]\otimes\cdots\otimes[a_n])$ in the form $b+h+\ell$ with $b\in B$, $h\in H$, and $\ell\in L$, we can read off equations (A.4.5) and (A.4.6) immediately. (For (A.4.6), since $n\geq 2$, our choice of $L'\subseteq L$ guarantees that $[a_1|\cdots|a_n]$ is in L.)

Now we prove (A.4.2) using induction on n. When $n=2$, λ_2 is just multiplication in $\Omega B_\infty^{\mathrm{aug}}A$, and multiplication in $\Omega B_\infty^{\mathrm{aug}}A$ is concatenation of tensors, so

$$\lambda_2([a_1]\otimes[a_2])=[a_1]\otimes[a_2].$$

The formula for d_1 from (A.4.1) yields the equation

$$d_1[a_1|a_2]=(-1)^{-1+\deg a_1}[a_1]\otimes[a_2].$$

This verifies (A.4.2) when $n=2$.

Now we proceed to the inductive step. We assume that equations (A.4.2)–(A.4.6) are valid for all $s<n$. The map λ_n is defined recursively as

$$\lambda_n=\sum_{s=1}^{n-1}(-1)^{s+1}\lambda_2\left(G\lambda_s\otimes G\lambda_{n-s}\right),$$

where $G\lambda_1=-\mathrm{id}$ and $\lambda_2=$ (multiplication). The left side of (A.4.2) is

$$\begin{aligned}\lambda_n([a_1]&\otimes\cdots\otimes[a_n])\\&=\sum_{s=1}^{n-1}(-1)^{s+1}\lambda_2\left(G\lambda_s\otimes G\lambda_{n-s}\right)([a_1]\otimes\cdots\otimes[a_n]).\end{aligned}$$

Using the Koszul sign convention and (A.4.6), we get

$$=\sum_{s=1}^{n-1}(-1)^{b_s}[a_1|\cdots|a_s]\otimes[a_{s+1}|\cdots|a_n],$$

where

$$b_s=s+1+\sum_{i\leq s}(n-i+1)\deg a_i+\sum_{i=s+1}^{n-1}(n-i)\deg a_i.$$

The formula for d_1 gives

$$d_1[a_1|\cdots|a_n]=\sum_{s=1}^{n-1}(-1)^{c_s}[a_1|\cdots|a_s]\otimes[a_{s+1}|\cdots|a_n],$$

where

$$c_s=\sum_{i\leq s}(-1+\deg a_i)=-s+\sum_{i\leq s}\deg a_i.$$

The ratio of the signs is

$$\begin{aligned}\mathrm{parity}(b_s+c_s)&=\mathrm{parity}(1+\sum_{i\leq s}(n-i)\deg a_i+\sum_{i=s+1}^{n-1}(n-i)\deg a_i)\\&=\mathrm{parity}(1+\sum_{i<n}(n-i)\deg a_i)=\mathrm{parity}(v_n),\end{aligned}$$

as desired. This finishes the induction, hence the verification of equation (A.4.2), hence the proof of the theorem. □

One consequence of Theorem A.4 is the following extension of Theorem 3.1. The difference between the theorem and its extension is in the choice of A_∞-algebra structures: in the theorem, the structure was required to come from a Merkulov model for the DG algebra A, while we can now use Theorem A.4 to show that the result depends only on the isomorphism type of the A_∞-algebra structure.

Corollary A.5. *Let A be a DG algebra. The result of Theorem 3.1 depends only on the isomorphism type of the A_∞-structure of HA.*

More precisely, choose an A_∞-algebra structure for HA so that there is an A_∞-algebra isomorphism $HA \to A$, as in Theorem 2.1. Fix $n \geq 3$. If $\alpha_1, \dots, \alpha_n \in HA$ are elements such that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined, then with b as in Theorem 3.1,

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \dots, \alpha_n \rangle.$$

A direct proof of this seems technically difficult: given an A_∞-isomorphism $f : (H, m_2, m_3, \dots) \to (H, m_2', m_3', \dots)$ with $f_1 = \mathrm{id}$, one would want to show that

$$m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) - m_n'(\alpha_1 \otimes \cdots \otimes \alpha_n) \tag{A.5.1}$$

is always in the indeterminacy of the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$. Although one can use the morphism identity **MI(n)** to rewrite this difference in terms of the morphism f, indeterminacy computations for Massey products are notoriously difficult (and often seem to lead to errors), so we use the following, less direct, proof. (And thus one sees as a consequence that the expression (A.5.1) *is* in fact in the indeterminacy of the Massey product.)

Proof. First note that if $f : A \to A'$ is a DG algebra map and $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined in HA, then $\langle f(\alpha_1), \dots, f(\alpha_n) \rangle$ is defined in HA', and

$$f(\langle \alpha_1, \dots, \alpha_n \rangle) \subseteq \langle f(\alpha_1), \dots, f(\alpha_n) \rangle.$$

(The map f will take any defining system for $\langle \alpha_1, \dots, \alpha_n \rangle$ to a defining system for $\langle f(\alpha_1), \dots, f(\alpha_n) \rangle$.)

Now fix $n \geq 3$ and elements $\alpha_1, \dots, \alpha_n \in HA$ such that the Massey product $\langle \alpha_1, \dots, \alpha_n \rangle$ is defined. Suppose that $H = (H, m_2, m_3, \dots)$ is an A_∞-algebra structure on HA such that there is an A_∞-isomorphism $H \to A$. We want to show that

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \dots, \alpha_n \rangle.$$

Consider the following diagram:

$$\begin{tikzcd}
H \arrow[r, "\sim"] \arrow[d, "\mathrm{Mer}"', "\sim"] & A \arrow[d, bend right, "\sim"] \\
\Omega B_\infty^{\mathrm{aug}} H \arrow[r, "\sim", "\mathrm{DG}"'] & \Omega BA \arrow[u, bend right, "\mathrm{DG}"']
\end{tikzcd}$$

The maps marked "DG" are DG maps, and the rest are A_∞-algebra maps. According to Theorem A.4, we may assume that H is given its A_∞-algebra structure via a Merkulov model on $\Omega B_\infty^{\mathrm{aug}} H$, and we have indicated this by labeling the appropriate arrow with "Mer". Since Ω and B are adjoint functors, there is a natural DG algebra map $\Omega BA \to A$; this is the DG map going up. All of the maps in the diagram are quasi-isomorphisms; indeed, the map $H \to A$ lifts the identity on H, and the map $\Omega B_\infty^{\mathrm{aug}} H \to \Omega BA$ induces the identity map $\mathrm{id} : H \to H$.

The map of DG algebras $\Omega B_\infty^{\mathrm{aug}} H \to A$ induces an inclusion of Massey products, as noted above. That is, with a slight abuse of notation (since the induced map on homology is the identity),

$$\langle \alpha_1, \dots, \alpha_n \rangle_{H(\Omega B_\infty^{\mathrm{aug}} H)} \subseteq \langle \alpha_1, \dots, \alpha_n \rangle_{H(A)}.$$

Since H comes from a Merkulov model on $\Omega B_\infty^{\mathrm{aug}} H$, it satisfies Theorem 3.1, so

$$(-1)^b m_n(\alpha_1 \otimes \cdots \otimes \alpha_n) \in \langle \alpha_1, \dots, \alpha_n \rangle_{H(\Omega B_\infty^{\mathrm{aug}} H)}.$$

This finishes the proof. □

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