1. [**15 points total**] Evaluate the following indefinite integrals. Show all work.

(a) [**5 points**] $\displaystyle\int \frac{3+\ln(t)}{t}\,dt.$

Write as two integrals. In the second, let $u = \ln(t)$.

$$\begin{aligned}\int \frac{3+\ln(t)}{t}\,dt &= \int \frac{3}{t}\,dt + \int \frac{\ln(t)}{t}\,dt \\ &= 3\ln(t) + \int u\,du \\ &= 3\ln(t) + \frac{1}{2}u^2 + C \\ &= 3\ln(t) + \frac{1}{2}[\ln(t)]^2 + C\end{aligned}$$

(b) [**5 points**] $\displaystyle\int \frac{1}{x(9-x^2)}\,dx.$

Use partial fractions.

$$\begin{aligned}\frac{1}{x(9-x^2)} &= \frac{1}{x(3-x)(3+x)} \\ &= \frac{(1/9)}{x} + \frac{(1/18)}{3-x} + \frac{(-1/18)}{3+x}\end{aligned}$$

Substituting,

$$\begin{aligned}\int \frac{1}{x(9-x^2)}\,dx &= \int \left[\frac{(1/9)}{x} - \frac{1/18}{x-3} - \frac{1/18}{x+3}\right] dx \\ &= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|x-3| - \frac{1}{18}\ln|x+3| + C \\ &= \frac{1}{18}\ln\left|\frac{x^2}{x^2-9}\right| + C\end{aligned}$$

(c) [**5 points**] $\displaystyle\int \frac{x+4}{x^2+4\,x+13}\,dx.$

First separate into $\displaystyle\int \frac{x+2}{x^2+4\,x+13}\,dx + \int \frac{2}{x^2+4\,x+13}\,dx$

To solve $\displaystyle\int \frac{x+2}{x^2+4\,x+13}\,dx$, *set* $u = x^2+4\,x+13$ *to get*

$$\begin{aligned}\int \frac{x+2}{x^2+4\,x+13}\,dx &= \int \frac{\frac{1}{2}\,du}{u}\\ &= \frac{1}{2}\ln|u| + C_1\\ &= \frac{1}{2}\ln\left(x^2+4\,x+13\right) + C_1\end{aligned}$$

Note that the argument of the log is always positive so we can drop the absolute value.

To solve $\displaystyle\int \frac{2}{x^2+4\,x+13}\,dx$, *complete the square in the denominator.*

$$\begin{aligned}\int \frac{2}{x^2+4\,x+13}\,dx &= \int \frac{2}{(x+2)^2+9}\,dx\\ &= \frac{1}{9}\int \frac{2}{\left(\frac{x+2}{3}\right)^2+1}\,dx\\ &= \frac{2}{3}\int \frac{1}{u^2+1}\,du \qquad \textit{setting } u = \frac{x+2}{3}\\ &= \frac{2}{3}\tan^{-1}u + C_2\\ &= \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C_2\end{aligned}$$

Thus the complete answer is $\displaystyle\frac{1}{2}\ln\left(x^2+4\,x+13\right) + \frac{2}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C$

2. [**15 points total**] Evaluate the following definite integrals. Show all work. Give your answers in exact form (not decimal approximations).

(a) [**5 points**] $\displaystyle\int_1^e x\ln x\,dx.$

By parts. Let $u = \ln(x)$ *and* $dv = x\,dx$. *Find that*

$$\begin{aligned}\int_1^e x\ln x\,dx &= \frac{1}{2}\,x^2\ln(x)\Big|_1^e - \frac{1}{2}\int_1^e x\,dx\\ &= \frac{1}{2}e^2 - \frac{1}{2}\left(\frac{1}{2}x^2\right)\Big|_1^e\\ &= \frac{1}{4}(e^2+1)\end{aligned}$$

(b) [**5 points**] $\displaystyle\int_{2\sqrt{2}}^{4} \frac{1}{y^2\sqrt{y^2-4}}\, dy$

Use the trig substitution $y = 2\sec\theta$*. Therefore,* $d\theta = 2\sec\theta\tan\theta\, d\theta$*. Substituting,*

$$\begin{aligned}\int_{2\sqrt{2}}^{4} \frac{1}{y^2\sqrt{y^2-4}}\, dy &= \int_{\pi/4}^{\pi/3} \frac{2\sec\theta\tan\theta}{4\sec^2\theta\sqrt{4\sec^2\theta-4}}\, d\theta \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/3} \frac{d\theta}{\sec\theta} \\ &= \frac{1}{4}\int_{\pi/4}^{\pi/3} \cos\theta\, d\theta \\ &= \frac{1}{4}\sin\theta\Big|_{\pi/4}^{\pi/3} \\ &= \frac{1}{8}(\sqrt{3}-\sqrt{2})\end{aligned}$$

(c) [**5 points**] $\displaystyle\int_{1}^{4} \frac{t}{2+\sqrt{t}}\, dt.$

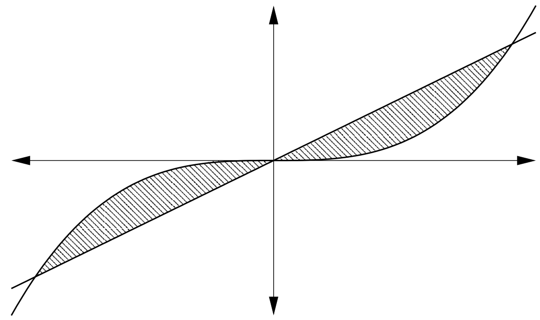
Let $u = \sqrt{t}$*. This implies that* $t = u^2$ *and* $dt = 2u\, du$*. Substituting, obtain*

$$\int_{1}^{4} \frac{t}{2+\sqrt{t}}\, dt = 2\int_{1}^{2} \frac{u^3}{2+u}\, du$$

Now let $z = 2+u$*. Obtain*

$$\begin{aligned}2\int_{1}^{2} \frac{u^3}{2+u}\, du &= 2\int_{3}^{4} \frac{(z-2)^3}{z}\, dz \\ &= 2\int_{3}^{4} \left(\frac{z^3-6z^2+12z-8}{z}\right) dz \\ &= 2\int_{3}^{4} \left(z^2-6z+12-\frac{8}{z}\right) dz \\ &= 2\left(\frac{1}{3}z^3-3z^2+12z-8\ln(z)\right)\Big|_{3}^{4} \\ &= 2\left(\frac{64}{3}-48+48-16\ln(2)\right)-2\Big(9-27+36-8\ln(3)\Big) \\ &= \frac{20}{3}+16\ln(3)-32\ln(2)\end{aligned}$$

3. [**8 points**] Find the total area of the two pieces of the region between $y = 4x$ and $y = x^3$.



$4x = x^3 \qquad x = 0, 2, -2$

$$\int_{-2}^{0} x^3 - 4x\ dx - \int_{0}^{2} x^3 - 4x\ dx$$

$$\int x^3 - 4x\ dx = \frac{1}{4}\,x^4 - 2\,x^2 + C$$

$$\left(\frac{1}{4}\,x^4 - 2\,x^2\right)\Bigg|_{-2}^{0} - \left(\frac{1}{4}\,x^4 - 2\,x^2\right)\Bigg|_{0}^{2} = 8$$

4. [**8 points**] Find $y(x)$ where $\dfrac{dy}{dx} = x\,e^{x+y}$ and $y(0) = -1$.

$$\int e^{-y}\,dy = \int x\,e^{x}\,dx$$

$$\int e^{-y}\,dy = -e^{-y} + C_1$$

$$\int x\,e^{x}\,dx = xe^{x} - e^{x} + C_2$$

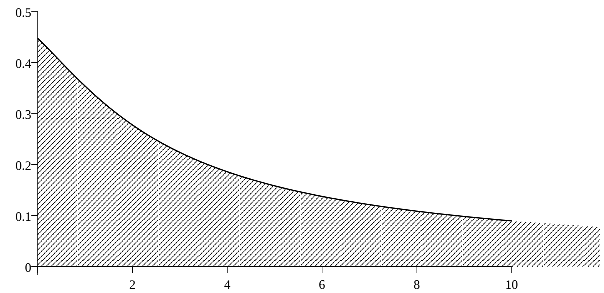
$$y = -\ln\Big((1-x)e^{x} + C\Big)$$

$$y = -\ln\Big((1-x)e^{x} + e - 1\Big)$$

5. [**10 points total**] The semi-infinite region bounded below by $y = 0$, on the left by $x = 0$, and above by the curve

$$y = \frac{1}{\sqrt{x^2 + 5x + 4}}$$

is rotated about the x-axis to form a solid of revolution. What is the volume of this solid?



$$\int_0^\infty \frac{\pi}{x^2 + 5x + 4}\, dx$$

$$\pi \int_0^\infty \frac{1/3}{x+1} - \frac{1/3}{x+4}\, dx$$

$$\frac{\pi}{3} \lim_{b\to\infty} \int_0^b \frac{1}{x+1} - \frac{1}{x+4}\, dx$$

$$\frac{\pi}{3} \lim_{b\to\infty} \big(\ln(x+1) - \ln(x+4)\big)\Big|_0^b$$

$$\frac{\pi}{3} \lim_{b\to\infty} \ln\left(\frac{x+1}{x+4}\right)\Bigg|_0^b$$

$$\frac{\pi}{3} \lim_{b\to\infty} \left[\ln\left(\frac{b+1}{b+4}\right) - \ln(1/4)\right]$$

$$\frac{\pi}{3} \ln(4)$$

6. [**10 points total**] An island is populated with 3000 birds. This population of birds has a relative growth rate of $k = 2\%$ per year. The birds are hunted at the constant rate of 100 birds per year.

(a) [**3 points**] Write a differential equation with initial condition for the bird population $y(t)$ after t years.

$$\frac{dy}{dt} = .02\,y - 100\,, \qquad y(0) = 3000$$

(b) [**4 points**] Solve the differential equation you found in part (a) to find an exact formula for $y(t)$. Show your steps clearly.

$$\frac{dy}{y - 5000} = .02\,dt\,, \qquad \ln|y - 5000| = .02\,t + C\,, \qquad y - 5000 = \pm e^{C}\,e^{.02\,t} = C_2\,e^{.02\,t}$$

$$y(0) = 3000 \quad \Rightarrow \quad C_2 = -2000\,, \qquad y(t) = 5000 - 2000\,e^{.02\,t}$$

(c) [**3 points**] At what time t will the bird population equal zero?

$$y(t) = 0 \quad \Rightarrow \quad 5000 = 2000\,e^{.02\,t} \quad \Rightarrow \quad 2.5 = e^{.02\,t} \quad \Rightarrow \quad t = 50\ln(2.5) \approx 45.8 \textit{ years}$$

7. [**10 points total**] Consider the function $f(x) = 2\sqrt{x}$.

(a) [**5 points**] Write down an integral formula for the arclength of the curve $y = f(x)$, for $1 \le x \le 3$. DO NOT EVALUATE THE INTEGRAL.

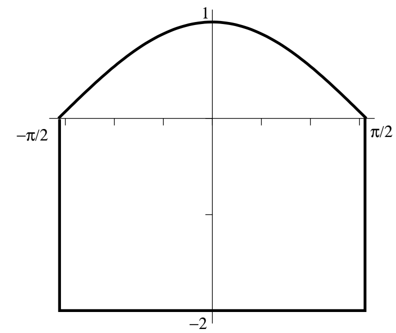
$$\frac{dy}{dx} = 2 \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{\sqrt{x}}, \quad \Rightarrow \quad \text{Length} = \int_1^3 \sqrt{1 + \frac{1}{x}}\, dx$$

(b) [**5 points**] Use Simpson's Rule with $n = 6$ to find an approximate value for the integral in part (a). Show clearly what you are doing. Give your answer in exact form (not a decimal approximation).

$$\int_1^3 g(x)\, dx \approx \tfrac{1/3}{3}\Big(g(1) + 4\,g(\tfrac{4}{3}) + 2\,g(\tfrac{5}{3}) + 4\,g(2) + 2\,g(\tfrac{7}{3}) + 4\,g(\tfrac{8}{3}) + g(3)\Big), \qquad g(x) = \sqrt{1 + \tfrac{1}{x}}$$

$$\text{Length} \approx \tfrac{1}{9}\Big(\sqrt{2} + 4\sqrt{\tfrac{7}{4}} + 2\sqrt{\tfrac{8}{5}} + 4\sqrt{\tfrac{3}{2}} + 2\sqrt{\tfrac{10}{7}} + 4\sqrt{\tfrac{11}{8}} + \sqrt{\tfrac{4}{3}}\Big)$$

8. [**8 points**] Find the centroid (center of mass) of the region bounded above by $y = \cos x$ and below by $y = -2$, between $x = -\pi/2$ and $x = \pi/2$. (See the picture.)



$$A = \int_{-\pi/2}^{\pi/2} \cos(x) + 2 \; dx$$
$$= \sin(x) + 2x \Big|_{-\pi/2}^{\pi/2}$$
$$= 2 + 2\pi$$

$$M_x = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \cos^2(x) - 4 \; dx$$
$$= \frac{1}{8}\sin(2x) - \frac{7}{4}x \Big|_{-\pi/2}^{\pi/2}$$
$$= -\frac{7}{4}\pi$$

so $\bar{y} = \frac{M_x}{A} = -\frac{7\pi}{8 + 8\pi}$
and $\bar{x} = 0$ *by symmetry.*

9. [**6 points**] Find $F''(1)$ where $F(x) = \int_1^{x+3\ln x} e^t \, dt$

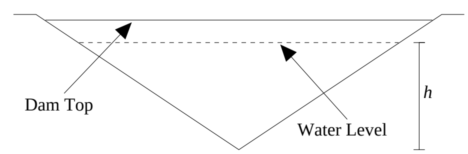
$F'(x) = e^{x+3\ln x} \cdot \left(1 + \frac{3}{x}\right)$ *by the Fundamental Theorem of Calculus and the Chain Rule*

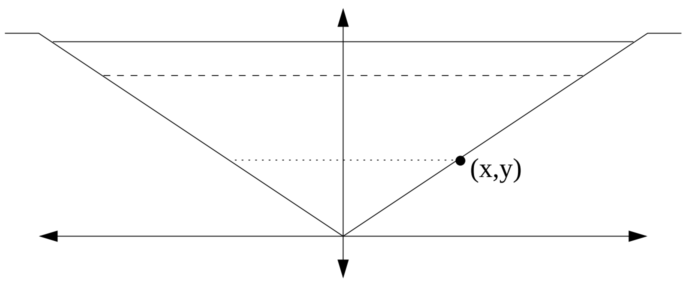
$F''(x) = e^{x+3\ln(x)}\left(1 + \frac{3}{x}\right)^2 - e^{x+3\ln(x)}\left(\frac{3}{x^2}\right)$ *by the Product Rule*

This simplifies to $F''(x) = xe^x\left(x^2 + 6x + 6\right)$

so $F''(1) = 13e$

10. [**10 points**] A small dam has the shape of the triangular region between $y = \frac{1}{2}|x|$ and $y = 4$, where distances are measured in meters. The dam can safely withstand a force of 180,000N. What is the maximum water level h that the dam can hold? (Use here 1000kg/m^3 as the density of water and for simplicity $g \approx 10\text{m/sec}^2$ as the gravitational constant.)





The right profile of the dam is $y = \frac{1}{2}x$ *so a horizontal slice at height* y *has length* $2x = 4y$.

$$\textit{Total Force on the dam } = \int_0^h 10 \cdot 1000 \cdot (h - y) \cdot 4y \; dy$$

$$= \frac{20000}{3} h^3 \textit{ Newtons}$$

Solve $\frac{20000}{3} h^3 = 180,000$ *to get* $h = 3$ *meters.*