DESCENT TECHNIQUES IN MODULAR REPRESENTATION THEORY

ABSTRACT. These are notes of talks given by Paul Balmer at the Seattle π-school in the summer of 2012. This series of four talks gives an introduction to some of the results of his paper [Bal12]. More details, the proofs, and motivational remarks can be found in that paper. The notes were taken by Moritz Groth who apologizes in advance for all typos.

1. LECTURE

In these lectures we discuss some aspects of *modular representation theory.* Thus, we are interested in the following setup; let G be a finite group, let k be a field of characteristic p, and let us impose the condition that p divides the order $|G|$ of the group. From the perspective of representation theory this is the more interesting case since we have:

$$kG \text{ not semi-simple} \qquad \Longleftrightarrow \qquad \mathit{char}(k) \text{ divides the order of } G$$

Guided by the general philosophy of *tensor triangular geometry* [Bal10], let us motivate the content of these lectures by drawing some analogies between algebraic geometry and modular representation theory.

We begin by establishing some notation (and recalling some general philosophy) from algebraic geometry. Let $X = (X, \mathcal{O}_X)$ be a noetherian, separated scheme (e.g., X could be the Zariski spectrum of a noetherian ring or an algebraic variety). Moreover, let

$$\mathcal{C}(X) \quad = \quad D(X)$$

be the **derived category** of X. Thus, $\mathcal{C}(X)$ is obtained from the category of quasi-coherent $\mathcal{O}_X$-modules by universally inverting the quasi-isomorphisms. It is well-known that $\mathcal{C}(X)$ is a triangulated category and that, moreover, it can be endowed with the (derived) tensor product turning it into a *tensor triangular category.* Now, let $j\colon U \subset X$ be an open subscheme in X. The induced restriction functor at the level of derived categories

$$j^*\colon D(X) \to D(U)$$

is a (categorical) localization functor (in the sense of localization theory of triangulated categories). A similar result holds true if one restricts to compact objects on both sides and passes to an idempotent completion.

Let us give a more specific example of an open subscheme in the affine case. Thus, let us consider a localization of rings $R \to R[1/s]$. The corresponding localization functor at the level of derived categories

$$D(R) \to D(R[1/s])$$

is induced by the **extension of scalar** functor $R[1/s] \otimes_R -\colon R\text{-}\mathsf{Mod} \to R[1/s]\text{-}\mathsf{Mod}$.

Let us summarize this recap by the following punchline. In algebraic geometry, in order to understand interesting 'global' objects, e.g., the derived category $D(X)$ of a scheme, one often tries to first understand them locally on opens. This can be meant in the classical sense or, more

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generally, in the sense of a Grothendieck topology. In the case mentioned above we saw that the passage to the local situation is provided by special instances of extension of scalar functors.

Using the notation of the first paragraph we now stick to modular representation theory and try to apply the above pattern to it. Let

$$\mathcal{C}(G) \quad = \quad kG\text{-}\mathsf{Stab}$$

be the **stable category** of all kG-modules. The upper case letter 'S' in Stab (and, similarly, in Mod and Proj) indicates that we do not impose any finiteness conditions on the modules. Thus, by the very definition, $\mathcal{C}(G)$ is obtained from the category kG-Mod of all kG-modules by dividing out the ideal of morphisms which factor through projective modules:

$$\mathcal{C}(G) \quad = \quad kG\text{-}\mathsf{Mod}/kG\text{-}\mathsf{Proj}$$

The objects of this category are just the kG-modules while morphisms are equivalence classes of morphisms in kG-Mod. Two parallel morphisms $f, f'\colon M \to M'$ are equivalent if and only if their difference factors through a projective module. It is easy to see that this defines a category but it can be canonically endowed with more structure. Using the diagonal G-action, we see that the tensor product of two kG-modules is naturally again a kG-module. One checks that we obtain an induced tensor product functor at the level of $\mathcal{C}(G)$. Moreover, $\mathcal{C}(G)$ can be naturally turned into a triangulated category so that we actually have a *tensor triangular category*. There are variants of this obtained by instead setting:

$$\mathcal{C}(G) \quad = \quad kG\text{-}\mathsf{Mod} \qquad \text{or} \qquad \mathcal{C}(G) \quad = \quad D(kG\text{-}\mathsf{Mod})$$

Here, $D(-)$ of course denotes the formation of derived categories.

In each case, given such a finite group G, we assign to it an interesting category $\mathcal{C}(G)$. Now, following the philosophy of algebraic geometry, in order to understand these categories $\mathcal{C}(G)$ we would like to first study them 'locally'. Here, this means that we consider subgroups $i\colon H \to G$ and study the corresponding restriction of scalar functors:

$$i^*\colon \mathcal{C}(G) \to \mathcal{C}(H)$$

Contrary to the situation in algebraic geometry, it turns out that –except in certain trivial cases– these restriction functor are *not* given by localization functors. But, motivated by the corresponding results in algebraic geometry, we might wonder whether these functors are given by certain extension of scalar functors. This turns out to be the case as we will discuss below.

Let us recall that a monoidal category $\mathcal{C}$ is a category $\mathcal{C}$ together with a tensor product $\otimes\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and a unit object $\mathbb{S} \in \mathcal{C}$ (see e.g., [ML98]). Moreover, there are specified coherence isomorphisms expressing that the tensor product is suitably associative and unital.

Definition 1.1. Let $(\mathcal{C}, \otimes, \mathbb{S})$ be a monoidal category. A **monoid object** in $\mathcal{C}$ is a triple (A, μ, η) consisting of an object $A \in \mathcal{C}$, a multiplication map $\mu\colon A \otimes A \to A$, and a unit map $\eta\colon \mathbb{S} \to A$ satisfying the following associativity and unitality conditions:

$$\begin{array}{ccccc} A\otimes(A\otimes A) & & \xrightarrow{\quad\cong\quad} & & (A\otimes A)\otimes A \\ {\scriptstyle 1\otimes\mu}\downarrow & & & & \downarrow{\scriptstyle \mu\otimes 1} \\ A\otimes A & \xrightarrow{\mu} & A & \xleftarrow{\mu} & A\otimes A \end{array} \qquad\qquad \begin{array}{ccccc} \mathbb{S}\otimes A & \xrightarrow{\eta\otimes 1} & A\otimes A & \xleftarrow{1\otimes\eta} & A\otimes\mathbb{S} \\ & {\scriptstyle\cong}\searrow & \downarrow{\scriptstyle\mu} & \swarrow{\scriptstyle\cong} & \\ & & A & & \end{array}$$

Example 1.2. (1) Rings, graded rings, and differential-graded rings are precisely the monoid objects in the category of abelian groups, graded abelian groups, and differential-graded abelian groups respectively.

(2) Given a category $\mathcal{C}$, then the category $\mathsf{End}(\mathcal{C})$ of endo-functors on $\mathcal{C}$ forms a monoidal category with respect to the composition and the identity functor. A **monad** (or a **triple**) on $\mathcal{C}$ is a monoid object in $\mathsf{End}(\mathcal{C})$.

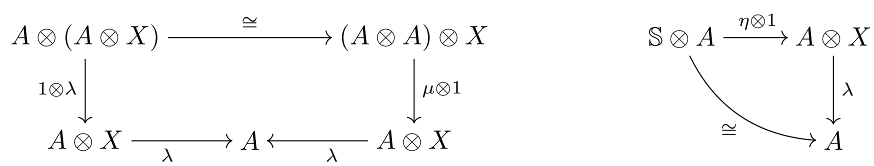
Exercise 1.3. Unravel the definition of a monad.

Monads are generalizations of monoid objects in the sense that every monoid object A in a monoidal category $\mathcal{C}$ gives rise to a monad $\mathbb{A}$ on $\mathcal{C}$ by setting:

$$\mathbb{A} = A \otimes - \colon \mathcal{C} \to \mathcal{C}$$

It is straightforward to use the structure maps of the monoid in order to endow $\mathbb{A}$ with the structure of a monad.

Definition 1.4. Let (A, μ, η) be a monoid object in a monoidal category $\mathcal{C}$. An A-**module** is a pair (X, λ) consisting of an object $X \in \mathcal{C}$ and an action map $\lambda \colon A \otimes X \to X$ such that the following diagrams commute:



With the obvious notion of morphisms this gives rise to the category $A\text{-}\mathsf{Mod}_{\mathcal{C}}$ of A-modules in $\mathcal{C}$.

Example 1.5. Let $\mathcal{C}$ be monoidal category and let A be a monoid object in $\mathcal{C}$. Given an arbitrary object $Y \in \mathcal{C}$ then we can form the **free** A-**module** $F_A(Y)$ generated by Y. The underlying object in $\mathcal{C}$ is $A \otimes Y$ while the action map is given by:

$$\lambda \colon A \otimes (A \otimes Y) \overset{\cong}{\to} (A \otimes A) \otimes Y \overset{\mu \otimes 1}{\to} A \otimes Y$$

With the obvious behavior on morphisms we obtain a free A-module functor $F_A \colon \mathcal{C} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}$. By forgetting structure, we obtain a forgetful functor $U_A \colon A\text{-}\mathsf{Mod}_{\mathcal{C}} \to \mathcal{C}$ in the other direction and one checks that we have an adjunction:

$$(F_A, U_A) \colon \mathcal{C} \rightleftarrows A\text{-}\mathsf{Mod}_{\mathcal{C}}$$

The functor F_A factors over the category $A\text{-}\mathsf{Free}_{\mathcal{C}}$ of free A-modules; this is the category with the same objects as $\mathcal{C}$ but where the morphism sets are given by:

$$\hom_{A\text{-}\mathsf{Free}_{\mathcal{C}}}(Y, Y') = \hom_{A\text{-}\mathsf{Mod}_{\mathcal{C}}}(F_A Y, F_A Y')$$

Thus, using the obvious fully faithful inclusion functor $A\text{-}\mathsf{Free}_{\mathcal{C}} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}$, the free A-module functor factors as

$$F_A \colon \mathcal{C} \to A\text{-}\mathsf{Free}_{\mathcal{C}} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}.$$

Similarly, we obtain a restricted free-forgetful adjunction as depicted in:

$$\mathcal{C} \rightleftarrows A\text{-}\mathsf{Free}_{\mathcal{C}}$$

The point of the next definition is that in the *separable* case the functor $A\text{-}\mathsf{Free}_{\mathcal{C}} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}$ is an equivalence up to an idempotent completion.

Definition 1.6. A monoid object A in a symmetric monoidal category is called **separable** if the multiplication map

$$\mu\colon A \otimes A \to A$$

has an A-bilinear section.

Let us unravel this definition a bit. Both objects A and $A \otimes A$ are modules over $A \otimes A^{op}$ by the obvious maps induced from μ. Note that we need the monoidal structure to be symmetric for both the formation of opposites and tensor products of monoid objects. Now, a A-bilinear section $\sigma\colon A \to A \otimes A$ is such a morphism in $\mathcal{C}$ such that $\mu \circ \sigma = 1_A$ and

$$(\mu \otimes 1) \circ (1 \otimes \sigma) \quad = \quad \sigma \circ \mu \quad = \quad (1 \otimes \mu) \circ (\sigma \otimes 1).$$

Example 1.7. Let $\mathcal{C} = \mathsf{Vect}_k$ be the monoidal category of vector spaces over a field k with the usual tensor product as monoidal structure. A monoid $A = K$ which happens to be a field is separable if and only if the field extension $\eta\colon k \to K$ is separable.

Example 1.8. More generally, given a finite étale ring extension $R \to A$ of commutative rings then the monoid A in R-Mod is separable.

Proposition 1.9. *Let A be a separable monoid in a symmetric monoidal category $\mathcal{C}$, then the (fully faithful) inclusion functor*

$$A\text{-}\mathsf{Free}_{\mathcal{C}} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}$$

is an equivalence up to direct summands.

Let us now return to the context of representation theory. Let G be a (finite) group, k be a commutative ring, and $H \subseteq G$ be a finite index subgroup. Let

$$A = A_H^G = k(G/H)$$

be endowed with the usual left G-action. Moreover, let $\mu\colon A \otimes A \to A$ be the k-bilinear extension of the map which sends $\gamma \otimes \gamma'$ to γ if $\gamma = \gamma' \in G/H$ and to 0 otherwise. The finite index assumption allows us to define a unit by forming the sum over a complete set of coset representatives. Thus, in each of the three contexts

$$\mathcal{C}(G) = kG\text{-}\mathsf{Mod}, \quad \mathcal{C}(G) = D(kG\text{-}\mathsf{Mod}), \quad \text{or} \quad \mathcal{C}(G) = kG\text{-}\mathsf{Stab}$$

we are given a monoid object and we have a corresponding free A-module functor F_A. In the last case, we of course make the modularity assumption that k is a field of characteristic p and that p divides the order of G. We then have the following result which can be described by the slogan:

'Restriction is extension'

Theorem 1.10. *In this notation, there is an equivalence of categories $\psi\colon \mathcal{C}(H) \to A\text{-}\mathsf{Mod}_{\mathcal{C}(G)}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{C}(G) & \xrightarrow{F_A} & \\ {\scriptstyle \mathsf{Res}_H^G}\downarrow & \searrow & \\ \mathcal{C}(H) & \xrightarrow[\psi]{\simeq} & A\text{-}\mathsf{Mod}_{\mathcal{C}(G)} \end{array}$$

Instead of giving a proof let us content ourselves by mentioning the key lemma going into the proof. More details can be found in [Bal12]. Thus, let us consider an adjunction

$$(F, G) \colon \mathcal{C} \rightleftarrows \mathcal{D}$$

and let us denote the unit and the counit by $\eta \colon 1_{\mathcal{C}} \to G \circ F$ and $\epsilon \colon F \circ G \to 1_{\mathcal{D}}$ respectively. The triangular identities of an adjunction imply that we obtain a monad $\mathbb{A} = G \circ F$ on $\mathcal{C}$ with structure maps given by $\eta \colon 1 \to \mathbb{A}$ and $\mu = G\epsilon F \to \mathbb{A} \circ \mathbb{A} \to \mathbb{A}$. To describe this situation we also say that the adjunction (F, G) *realizes the monad* $\mathbb{A}$.

Earlier in this lecture we recalled the concept of a module over a monoid in a monoidal category and also sketched the construction of free modules in that generality. The aim of the next exercise is to extend this to the context of monads.

Exercise 1.11. (1) Let $\mathbb{A}$ be a monad. Define a notion of a **module over the monad** $\mathbb{A}$ and morphisms between such modules in order to obtain the category $\mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$.

(2) Let A be a monoid in the monoidal category $\mathcal{C}$ and let $\mathbb{A} = A \otimes -$ be the associated monad. Check that there is an isomorphism of categories $A\text{-}\mathsf{Mod}_{\mathcal{C}} \cong \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$.

(3) Given a monad $\mathbb{A}$ on a category $\mathcal{C}$, define the notion of a **free module over the monad** $\mathbb{A}$ and the **free module functor** $F_{\mathbb{A}} \colon \mathcal{C} \to \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$.

(4) Define the category $\mathbb{A}\text{-}\mathsf{Free}_{\mathcal{C}}$ of free $\mathbb{A}$-modules.

(5) Show that every monad can be realized in at least two ways namely using suitable forgetful functors:

$$\mathcal{C} \rightleftarrows \mathbb{A}\text{-}\mathsf{Free}_{\mathcal{C}} \qquad \text{and} \qquad \mathcal{C} \rightleftarrows \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$$

The realization of the monad using the adjunction $\mathcal{C} \rightleftarrows \mathbb{A}\text{-}\mathsf{Free}_{\mathcal{C}}$ is called the **Kleisli construction** while the adjunction $\mathcal{C} \rightleftarrows \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$ is referred to as the **Eilenberg-Moore construction**. It is a classical fact from category theory that given a monad $\mathbb{A}$ then the Kleisli construction is the initial example of an adjunction realizing the given monad. Similarly, the Eilenberg-Moore construction is the terminal such example. In particular, given an arbitrary adjunction $(F, G) \colon \mathcal{C} \rightleftarrows \mathcal{D}$ then there are two comparison functors:

$$\mathbb{A}\text{-}\mathsf{Free}_{\mathcal{C}} \xrightarrow{K} \mathcal{D} \qquad \text{and} \qquad \mathcal{D} \xrightarrow{E} \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$$

The functor K is called the Kleisli comparison functor while the functor E is referred to as the Eilenberg-Moore comparison functor. Note that these functors are morphisms of adjunctions, i.e., they are compatible with both the respective left and right adjoint functors.

We can now close this lecture by mentioning the key lemma which is used to prove the above theorem.

Lemma 1.12. ***(Key lemma)***
Let $(F, G) \colon \mathcal{C} \rightleftarrows \mathcal{D}$ be an adjunction and assume that the adjunction counit $\epsilon \colon F \circ G \to 1_{\mathcal{D}}$ has a section (i.e., that there is a natural transformation $\xi \colon 1_{\mathcal{D}} \to F \circ G$ such that $\epsilon \circ \xi = 1$).

(1) *The induced monad $\mathbb{A} = GF$ on $\mathcal{C}$ is separable.*

(2) *The Kleisli and the Eilenberg-Moore comparison functors are equivalences up to direct summands.*

(3) *If we assume in addition that $\mathcal{C}$ and $\mathcal{D}$ are idempotent complete categories, then the Eilenberg-Moore comparison functor $E \colon \mathcal{D} \to \mathbb{A}\text{-}\mathsf{Mod}_{\mathcal{C}}$ is an equivalence of categories (i.e., the adjunction (F, G) is monadic).*

2. Lecture

Let G again be a finite group, k a field of positive characteristic p such that p divides the order $|G|$ of G. Moreover, let us denote by $\mathcal{C}(G)$ one of the following categories:

$$kG\text{-}\mathsf{Mod}, \qquad D(kG), \qquad \text{or} \qquad \mathsf{Stab}(kG)$$

Given a subgroup $H \to G$ we constructed a monoid object $A_H^G \in \mathcal{C}(G)$ such that $\mathcal{C}(H)$ is equivalent to A-$\mathsf{Mod}_{\mathcal{C}(G)}$. More precisely, we showed this equivalence to translate the restriction-induction adjunction $\mathcal{C}(G) \leftrightarrows \mathcal{C}(H)$ into the adjunction $\mathcal{C}(G) \leftrightarrows A_H^G$-$\mathsf{Mod}_{\mathcal{C}(G)}$. (Note that the induction functor Ind_H^G can be considered as a *coinduction* functor $Coind_H^G$ since we consider a finite index subgroup.) The aim today is to use descent techniques in order to 'identify $\mathcal{C}(G)$ inside of A_H^G-$\mathsf{Mod}_{\mathcal{C}(G)}$'.

Let us begin by considering a general symmetric monoidal category $\mathcal{C}$ together with a monoid object A in $\mathcal{C}$. The aim is to identify $\mathcal{C}$ in A-$\mathsf{Mod}_{\mathcal{C}}$. This can only be possible if the functor $A \otimes -\colon \mathcal{C} \to A$-$\mathsf{Mod}_{\mathcal{C}}$ is faithful. Independently of this assumption, let us define the category

$$\mathsf{Desc}_{\mathcal{C}}(A)$$

of **descent data** with respect to A. An object in this category is a pair (M, γ) consisting of $M \in A$-$\mathsf{Mod}_{\mathcal{C}}$ together with a **gluing isomorphism** $\gamma\colon A \otimes M \to M \otimes A$ in $(A \otimes A)$-$\mathsf{Mod}_{\mathcal{C}}$. This datum has to satisfy the following *cocycle condition* in $A^{\otimes 3}$-$\mathsf{Mod}_{\mathcal{C}}$:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\gamma_1 = 1\otimes\gamma} & A \otimes M \otimes A \\ & \searrow^{\gamma_2} & \downarrow{\scriptstyle \gamma_3=\gamma\otimes 1} \\ & & M \otimes A \otimes A \end{array}$$

Here, the morphism γ_2 is defined –again using the symmetry constraint τ of the monoidal structure– by the following composition:

$$A \otimes A \otimes M \xrightarrow{\tau\otimes 1} A \otimes A \otimes M \xrightarrow{1\otimes\gamma} A \otimes M \otimes A \xrightarrow{\tau\otimes 1} M \otimes A \otimes A$$

Given two such descent data (M, γ) and (M', γ') a morphism $(M, \gamma) \to (M', \gamma')$ is a morphism $f\colon M \to M'$ in A-$\mathsf{Mod}_{\mathcal{C}}$ which is compatible with the descent data:

$$\begin{array}{ccc} A \otimes M & \xrightarrow{1\otimes f} & A \otimes M' \\ {\scriptstyle\gamma}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\gamma'} \\ M \otimes A & \xrightarrow[f\otimes 1]{} & M' \otimes A \end{array}$$

We can define a **comparison functor** $Q\colon \mathcal{C} \to \mathsf{Desc}_{\mathcal{C}}(A)$ which sends an object $X \in \mathcal{C}$ to the descent datum $Q(X) = (A \otimes X, \gamma)$ with $\gamma = 1 \otimes \tau\colon A \otimes A \otimes X \to A \otimes X \otimes A$. On morphisms we simply set $Q(f) = 1 \otimes f$.

Exercise 2.1. (1) For $X \in \mathcal{C}$ the pair $Q(X)$ in fact defines a descent datum with respect to A.
(2) The above assignments define a functor $Q\colon \mathcal{C} \to \mathsf{Desc}_{\mathcal{C}}(A)$.

Definition 2.2. Let $\mathcal{C}$ be a symmetric monoidal category and let A be a monoid object in $\mathcal{C}$. The monoid A **satisfies descent** if the descent functor $Q\colon \mathcal{C} \to \mathsf{Desc}_{\mathcal{C}}(A)$ is an equivalence.

Let us give an example from algebraic geometry motivating the terminology.

Example 2.3. Let R be a commutative ring such that its spectrum $Spec(R)$ is covered by basic opens $Spec(R) = U(s_1) \cup \ldots \cup U(s_n)$. Let us recall that $U(s) = \{\mathfrak{p} \mid s \notin \mathfrak{p}\}$. Let A be the product of the localizations:

$$A = R_{s_1} \times \ldots \times R_{s_n}$$

Then $A \otimes A = \Pi_{i,j} R_{s_i s_j}$ and the theory of **Zariski descent** tries to construct an R-module from an $(R_{s_1} \times \ldots \times R_{s_n})$-module endowed with some additional structure.

Before we state the following theorem let us recall that a monoid object A in a monoidal category is called **faithful** if the free module functor $\mathbb{A} = A \otimes -: \mathcal{C} \to A\text{-}\mathsf{Mod}_{\mathcal{C}}$ is faithful.

Theorem 2.4. *Let $\mathcal{C}$ be a symmetric monoidal category and let A be a monoid object in $\mathcal{C}$. If $\mathcal{C}$ is in addition an idempotent complete triangulated category, then A satisfies descent if and only if A is faithful.*

We want to have such a theorem in the context of modular representation theory: $\mathcal{C} = \mathcal{C}(G)$, $A = A_H^G = k(G/H)$. The unit of this monoid $\eta\colon k \to A$ is given by $1 \mapsto \sum_{\gamma \in G/H} \gamma$ where γ runs through a complete set of representatives. One can establish the following result.

Proposition 2.5. *In the above notation, the monoid A_H^G is faithful iff $[G : H]$ is a unit in k iff p does not divide $[G : H]$ iff H contains a p-Sylow subgroup of G.*

If we want to unfold the definition of $\mathsf{Desc}_{\mathcal{C}(G)}(A_H^G)$ then we need, in particular, explicit descriptions of $A_H^G \otimes A_H^G$ and of the threefold tensor product. More generally, given two subgroups H_1 and H_2 then for $A_{H_1}^G \otimes A_{H_2}^G$ we obtain:

$$A_{H_1}^G \otimes A_{H_2}^G \cong \bigoplus_{[t]} A_{H_1^t \cap H_2}^G$$

Here, given a subgroup K and a group element g we used the notation $K^g = g^{-1}Kg$. Moreover, the sum runs over a complete set of representatives of double cosets $H_1 t H_2$ and hence involves some choices. These choices become quite complicated, in particular, if it comes to the description of the threefold tensor product – as it shows up if one completely unravels the notion of a descent datum.

We do not pursue these lines any further but will take a different perspective. Instead of only considering the orbits of the group G we pass to the category of *all* G-sets. The role of representations of subgroups is then played by the more general representations of G-sets (see Example 2.10).

As a preparation for this shift of perspective let us recall the basic philosophy of **Grothendieck topologies**. Given a classical topological space X, then we have by the very definition inclusions of the open subspaces $U \to X$. Derived from this we have the notion of an open cover of X which is just a special collection $\{U_i \to X\}$ of such inclusions of opens. If we consider this from a more categorical perspective we see that:

(1) Opens in X are just certain maps with a fixed codomain.
(2) Covers are just collections of such maps.

Grothendieck abstracted from this situation and introduced the concept of a Grothendieck topology. In particular, the assumption that the morphisms in a cover are inclusions of subobjects is dropped in this more general notion! A central role in algebraic geometry is played by the Zariski topology. Given a commutative ring R, then an example of a **Zariski cover** of $Spec(R)$ is given by a map of the form $Spec(A) \to Spec(R)$ with A as in Example 2.3. In modular representation, we want to think of the inclusion of a subgroup $H \to G$ as a cover if the index of this subgroup is prime to p.

Given a G-set X and an element $x \in X$ then we write $St_G(x)$ for the stabilizer subgroup of x. In particular, given a map $f\colon X \to X'$ of G-sets then we have an inclusion of subgroups $St_G(x) \to St_G(f(x))$ for all $x \in X$.

Definition 2.6. Let $\mathcal{G} = G\text{-}\mathsf{Sets}$ be the category of finite G-sets and let p be a prime number. A collection of maps $\{\alpha_i\colon U_i \to X\}_{i\in I}$ is a **sipp-covering** (**s**tabilizer **i**ndex **p**rime to **p**) if for all $x \in X$ there is an element $u_i \in U_i$ for some $i \in I$ with $\alpha_i(u_i) = x$ and the index $[St_G(x) : St_G(u_i)]$ is prime to p.

Example 2.7.
(1) Let $K \to H \to G$ be inclusions of subgroups. Then the canonical map $G/K \to G/H$ is a sipp cover if and only if the index $[H : K]$ is prime to p.
(2) Given a subgroup $P \to G$, then the map $G/P \to * = G/G$ is a sipp cover if and only if P is a p-Sylow subgroup.

Theorem 2.8. *The collection of sipp-coverings defines a Grothendieck topology on $\mathcal{G}$, i.e., the following three properties hold true:*

(1) *Every isomorphism $U \overset{\cong}{\to} X$ is a cover.*
(2) *If $\{U_i \to X\}_{i\in I}$ is a cover and $X' \to X$ is a map in $\mathcal{G}$ then the collection $\{U_i \times_X X' \to X'\}_{i\in I}$ is a cover.*
(3) *If $\{U_i \to X\}_{i\in I}$ is a cover and if for all $i \in I$ we have a cover $\{V_{ij} \to U_i\}_{j\in J_i}$ then the collection $\{V_{ij} \to X\}_{i,j}$ is a cover.*

The point of having a Grothendieck topology on a category is that we can then talk about sheaves or stacks defined on them. Since the category $\mathcal{G}$ admits finite coproducts we can always replace a covering consisting of finitely many maps by a covering with a single morphisms only. Moreover, it can be checked that the topology is quasi-compact so that in order to check whether a presheaf is actually a sheaf it suffices to consider covers given by a single morphism.

In order to consider some interesting examples of presheaves (or prestacks) on $\mathcal{G}$ let us recall the following concept. Let $X \in \mathcal{G}$ then a **representation** V of X (over k) consists of k-modules V_x, $x \in X$ together with k-linear maps $V_g\colon V_x \to V_{gx}$ for all $x \in X$ and $g \in G$. These maps have to satisfy the obvious compatibility conditions:

$$V_1 = 1\colon V_x \to V_x \qquad \text{and} \qquad V_{hg}\colon V_x \overset{V_g}{\to} V_{gx} \overset{V_h}{\to} V_{hgx}$$

Exercise 2.9.
(1) Given a G-set X, define a category $\int X$ such that a representation of G (over k) 'is precisely the same thing as' a functor $\int X \to k\text{-}\mathsf{Mod}$. The category $\int X$ is called the **action groupoid** of X.
(2) Define a **morphism of representations** of G-sets as a natural transformation of functors $\int X \to k\text{-}\mathsf{Mod}$ and unravel the definition in more explicit terms.

Thus, associated to each G-set X we have the additive (actually abelian) **category** $\mathsf{Rep}(X)$ **of representations** of X (over k). The above exercise shows that we have an isomorphism of categories:

$$\mathsf{Fun}(\int X, k\text{-}\mathsf{Mod}) \cong \mathsf{Rep}(X)$$

Example 2.10. For the special G-set $X = G/H$ we have an equivalence of categories

$$\mathsf{Rep}(G/H) \overset{\simeq}{\to} kH\text{-}\mathsf{Mod}$$

which sends a representation V to its value at the coset H.

It is a nice exercise to verify this in elementary terms. From a more conceptual perspective it suffices to observe that the action groupoid $\int G/H$ is a *connected* groupoid and hence equivalent to the group of automorphism of any of its elements. But this group is easily identified with H in this case.

Now, the assignment $X \mapsto \mathsf{Rep}(X)$ is itself functorial in the G-set as will be confirmed in the following exercise.

Exercise 2.11.
(1) Given a map of $\alpha\colon X \to Y$ in $\mathcal{G}$ show that there is a natural choice for a restriction functor $\alpha^*\colon \mathsf{Rep}(Y) \to \mathsf{Rep}(X)$.
(2) Given two maps $X \overset{\alpha}{\to} Y \overset{\beta}{\to} Z$ in $\mathcal{G}$ then we have $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$. Moreover, we have $(1_X)^* = 1_{\mathsf{Rep}(X)}$.

To summarize this construction let us denote by Add the category whose objects are additive categories while morphisms are given by additive functors. The assignments

$$X \mapsto \mathsf{Rep}(X), \qquad X \mapsto D(\mathsf{Rep}(X)), \qquad \text{and} \qquad X \mapsto \mathsf{Stab}(\mathsf{Rep}(X))$$

define functors $\mathcal{G}^{op} \to \mathsf{Add}$, i.e., presheaves of (additive) categories. For the third assignment we have to assume that the ground ring k is a field so that the category of representation is actually a *Frobenius* abelian category and we can hence form the associated stable categories. In the next lecture we will use the topology introduced in this lecture and observe that these presheaves satisfy descent with respect to this topology. Thus these assignments define **stacks** ('sheafs of categories') **with respect to the sipp-topology**.

3. Lecture

Let us briefly recall what we did in the last lecture. Given a group G, a field k of positive characteristic p, and a finite index subgroup $H \to G$, we constructed a monoid object $A = A_H^G$ in $\mathcal{C}(G)$. Moreover, we saw that the comparison functor

$$Q\colon \mathcal{C}(G) \to \mathsf{Desc}_{\mathcal{C}(G)}(A)$$

is an equivalence if and only if the order $[G : H]$ is prime to p. However, unraveling the details about the descent category is quite a bit technical and involves non-canonical choices (Mackey isomorphisms) so that we preferred to change the perspective slightly. Instead of considering orbits only we passed to the category of all finite G-sets and introduced the sipp-topology on it.

In the last lecture we introduced the notion of a representation of a $\mathcal{G}$-set X. More precisely, we saw that there is an additive category $\mathsf{Rep}(X)$ of k-linear representations of X. The assignment $X \mapsto \mathsf{Rep}(X)$ is contravariant in X. Let us denote either of the following assignments

$$X \mapsto \mathsf{Rep}(X), \qquad X \mapsto D(\mathsf{Rep}(X)), \qquad \text{and} \qquad X \mapsto \mathsf{Stab}(\mathsf{Rep}(X))$$

by $C \mapsto \underline{\mathcal{C}}(X)$.

Example 3.1. If the G-set X is $X = G/H$ then we have an equivalence $\underline{\mathcal{C}}(G/H) \to \mathcal{C}(H)$ of categories.

In each of the three cases, we obtain a functor $\underline{\mathcal{C}}\colon \mathcal{G}^{op} \to \mathsf{Add}$ where Add denotes the category of additive categories and additive functors. The descent property translates into the following theorem. The aim of this lecture is to explain the notions showing up in this theorem and then to specialize to the case where the G-sets under consideration are again certain orbits.

Theorem 3.2. *The functor* $\underline{\mathcal{C}} \colon \mathcal{G}^{op} \to \mathsf{Add}$ *is a stack with respect to the sipp topology, i.e., for every sipp cover* $\mathcal{U} = (U \stackrel{\alpha}{\to} X)$ *the following functor is an equivalence:*

$$Q \colon \underline{\mathcal{C}}(X) \to \mathsf{Desc}_{\underline{\mathcal{C}}}(\mathcal{U})$$

Before we introduce the target category of the functor under consideration let us introduce some notation. Given a morphism $\alpha \colon U \to X$ in $\mathcal{G}$ let us denote by $U^{(2)}$ the pullback defined by the following square:

$$\begin{array}{ccc} U^{(2)} = U \times_X U & \xrightarrow{pr_2} & U \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ U & \xrightarrow[\alpha]{} & X \end{array}$$

Similarly, let us write $U^{(n)}$ for iterated such pullbacks with n factors. Moreover, let us denote by

$$pr_{ij} \colon U^{(3)} \to U^{(2)}$$

the projection morphism onto the i-th and the j-th component for $1 \leq i < j \leq 3$.

With this notation we can introduce the **category of descent data**

$$\mathsf{Desc}_{\underline{\mathcal{C}}}(\mathcal{U})$$

associated to a cover $\mathcal{U} = (U \to X)$ of $X \in \mathcal{G}$ with respect to the sipp topology. An object is a pair (W, γ) consisting of an object $W \in \underline{\mathcal{C}}(U)$ together with an isomorphism

$$\gamma \colon pr_2^*(W) \stackrel{\cong}{\to} pr_1^*(W)$$

in $\underline{\mathcal{C}}(U^{(2)})$. This isomorphism has to satisfy the following cocycle condition on $U^{(3)}$:

$$pr_{13}^*(\gamma) = pr_{12}^*(\gamma) \circ pr_{23}^*(\gamma)$$

Given two such descent data (W, γ) and (W', γ'), a morphism $(W, \gamma) \to (W', \gamma')$ is a morphism $f \colon W \to W'$ in $\underline{\mathcal{C}}(U)$ such that the following square commutes in $\underline{\mathcal{C}}(U^{(2)})$:

$$\begin{array}{ccc} pr_2^*(W) & \xrightarrow{pr_2^*(f)} & pr_2^*(W') \\ {\scriptstyle \gamma}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \gamma'} \\ pr_1^*(W) & \xrightarrow[pr_1^*(f)]{} & pr_1^*(W) \end{array}$$

Exercise 3.3. There is a natural candidate for a comparison functor $Q \colon \underline{\mathcal{C}}(X) \to \mathsf{Desc}_{\underline{\mathcal{C}}}(U)$. Define it and check that it is well-defined.

A proof of the above theorem will not be given here but can be found in [Bal12]. Instead let us specialize the theorem to deduce some statements about $\mathcal{C}(H)$. For this purpose let us consider the special case of a sipp-cover given by:

$$\mathcal{U} = (U = G/H \to X = G/G = *)$$

Using a Mackey formula we obtain an isomorphism in $\mathcal{G}$

$$U^{(2)} = G/H \times G/H \cong \bigsqcup_{[t]} G/(H^t \cap H)$$

where $[t]$ runs over a complete set of representatives of double cosets. If we want to express $U^{(3)}$ in terms of orbits only we obtain an even more complicated formula. Moreover, in both cases the isomorphisms are not canonical since they involve choices of complete sets of representatives. The passage to the category $\mathcal{G}$ of *all* finite G-sets allows us to avoid these choices and is in spirit with the Grothendieckian philosophy:

(1) allow all choices (=making no choice)
(2) trim the excess of information

In order to apply this to our situation let us again begin by introducing some notation. Let $K, H \leq G$ be subgroups of our given group and let $g \in G$ be such that $gKg^{-1} \leq H$ (i.e., K is **subconjugate** to H). Then there is a G-map

$$\beta_g \colon G/K \to G/H \colon xK \mapsto xg^{-1}H.$$

In this notation the above Mackey isomorphism from the right to the left is given on the summand indexed by t by

$$[z] \quad \mapsto \quad (\beta_t, \beta_1)([z]) = (\beta_t([z]), [z]).$$

Here, t again runs over a complete set of representatives. Instead of making such a choice of representatives, we now consider the maps

$$(\beta_g, \beta_1) \colon G/(H^g \cap H) \to G/H \times G/H$$

for *all* $g \in G$.

Theorem 3.4. *(about $\underline{\mathcal{C}} \colon \mathcal{G}^{op} \to \mathsf{Add}$)*
Let $H \leq G$ be a subgroup of index prime to p, let $W \in \underline{\mathcal{C}}(G/H)$, and let $H[g] = H^g \cap H$ for every $g \in G$. Given isomorphisms

$$\sigma_g \colon \beta_1^*(W) \xrightarrow{\cong} \beta_g^*(W)$$

in $\underline{\mathcal{C}}(G/H[g])$ (where $\beta_1,\ \beta_g \colon G/H[g] \to G/H$ exist since $H[g] \leq H$ and $gH[g]g^{-1} \leq H$) satisfying

(1) *If $g = h \in H$ then $\sigma_h = id$ $(H[h] = H, \beta_1 = id \overset{!}{=} \beta_h)$.*
(2) *If $g_1, g_2 \in G$ then in $\underline{\mathcal{C}}(G/H[g_1, g_2])$ (where $H[g_1, g_2] = H^{g_2 g_1} \cap H^{g_1} \cap H$) we have*

$$\beta_1^*(\sigma_{g_1 g_2}) \quad = \quad \beta_{g_1}^*(\sigma_{g_2})\beta_1^*(\sigma_{g_1})$$

where

$$\beta_1 \colon G/H[g_1, g_2] \to G/H[g_2 g_1], \quad \beta_{g_1} \colon G/H[g_1, g_2] \to G/H[g_2], \quad \textit{and} \quad \beta_1 \colon G/H[g_1, g_2] \to G/H[g_1].$$

Then there is an essentially unique pair (V, f) consisting of an object $V \in \underline{\mathcal{C}}(G/G)$ and an isomorphisms $f \colon \beta_1^*(V) \to W$ in $\underline{\mathcal{C}}(G/H)$ which is compatible with the σ_g.

We now want to translate the statement of the theorem from $\underline{\mathcal{C}}$ to $\mathcal{C}$. Given a subgroup $H \leq G$ then we have an equivalence $i_H^* \colon \underline{\mathcal{C}}(G/H) \to \mathcal{C}(H)$. If we have subgroups $K \leq H \leq G$ then we have a morphism $\beta_1 \colon G/K \to G/H$ in $\mathcal{G}$. The above equivalences are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathcal{C}}(G/H) & \xrightarrow[\simeq]{i_H^*} & \mathcal{C}(H) \\ \big\downarrow{\scriptstyle \beta_1^*} & & \big\downarrow{\scriptstyle \mathrm{Res}_K^H} \\ \underline{\mathcal{C}}(G/K) & \xrightarrow[i_K^*]{\simeq} & \mathcal{C}(K) \end{array}$$

There is a further compatibility in the context of subconjugated subgroups. If ${}^gK = gKg^{-1} \leq H$ then we have the morphism

$$\beta_g \colon G/K \to G/H \colon [x] \mapsto [xg^{-1}].$$

Moreover, the conjugation map

$$K \to {}^gK \colon k \mapsto {}^gk = gkg^{-1}$$

combined with the inclusion ${}^gK \leq H$ induces a **twisted restriction of scalar functor**

$${}^g\mathrm{Res}^H_K \colon H\text{-}\mathsf{Set} \to {}^gK\text{-}\mathsf{Set} \to K\text{-}\mathsf{Set}.$$

Thus, given a set W with an H-action then ${}^g\mathrm{Res}^H_K(W)$ is the same set endowed with the K-action given by $k \cdot w = {}^gk \cdot w$. A similar reasoning applies if we consider objects in arbitrary categories endowed with a group action so that also in that general context we obtain twisted restriction of scalar functors. Now, our equivalences $i^*_H \colon \underline{\mathcal{C}}(G/H) \to \mathcal{C}(H)$ are compatible with subconjugated subgroups in the following sense. Given ${}^gK \leq H$ then the following diagram commutes up to a natural isomorphism $\omega^{(g)}$:

$$\begin{array}{ccc}
\underline{\mathcal{C}}(G/H) & \xrightarrow[\simeq]{i^*_H} & \mathcal{C}(H) \\
{\scriptstyle \beta^*_g}\downarrow & \Nearrow & \downarrow{\scriptstyle {}^g\mathrm{Res}^H_K} \\
\underline{\mathcal{C}}(G/K) & \xrightarrow[i^*_K]{\simeq} & \mathcal{C}(K)
\end{array}$$

Exercise 3.5. Define the natural isomorphism $\omega^{(g)}$.

Remark 3.6. (1) If $h \in H$, then there is an iso $h \cdot (-) \colon \mathrm{Res}^H_K(W) \to {}^h\mathrm{Res}^H_K(W) \colon w \mapsto h \cdot w$.
(2) If $W \in \mathcal{C}(H)$ lies in the image of $\mathcal{C}(G) \to \mathcal{C}(H)$ then there are necessarily isomorphisms:

$$\mathrm{Res}^H_{H[g]}(W) \to {}^g\mathrm{Res}^H_{H[g]}(W)$$

With this preparation we then obtain the following theorem about

$$\mathcal{C}(H) = kH\text{-}\mathsf{Mod}, \quad \mathcal{C}(H) = D(kH\text{-}\mathsf{Mod}), \quad \text{or} \quad \mathcal{C}(H) = kH\text{-}\mathsf{Stab}$$

which allows us to construct G-representations out of H-representations with additional data. The more interesting cases of the theorem are in the derived or in the stable context.

Theorem 3.7. *Let $W \in \mathcal{C}(H)$ and let us be given an isomorphisms $\sigma_g \colon Res^H_{H[g]}(W) \to {}^g Res^H_{H[g]}(W)$ for every $g \in G$ such that*

(1) *For all $g = h \in H$ we have $\sigma_h = h \cdot (-)$ in $\mathcal{C}(H)$.*
(2) *For all $g_1,\, g_2 \in G$ the following diagram commutes in $\mathcal{C}(H[g_1, g_2])$:*

$$\begin{array}{ccc}
Res^H_{H[g_2,g_1]}(W) & \xrightarrow{Res^{H[g_2g_1]}_{H[g_2,g_1]}(\sigma_{g_2g_1})} & {}^{g_2g_1}Res^H_{H[g_2,g_1]}(W) \\
{\scriptstyle Res^{H[g_1]}_{H[g_2,g_1]}(\sigma_{g_1})}\searrow & & \nearrow{\scriptstyle {}^{g_1}Res^{H[g_2]}_{H[g_2,g_1]}(\sigma_{g_2})} \\
& {}^{g_1}Res^H_{H[g_2,g_1]}(W) &
\end{array}$$

Then there is an essentially unique pair (V, f) consisting of an object $V \in \mathcal{C}(G)$ and an isomorphism $f \colon Res^G_H(V) \to W$ in $\mathcal{C}(H)$ which is compatible with the σ_g.

4. Lecture

We again begin by describing our setup. Let G be a finite group and k a field of positive characteristic p dividing the order of the group G. For a subgroup $H \to G$ let us set $\mathcal{C}(H) = kH\text{-}\mathsf{Stab}$ and this category is what we are ultimately interested in.

In order to study this category we generalized slightly by considering the site $\mathcal{G} = G\text{-}\mathsf{Set}$ of all finite G-sets with the sipp-topology. On this site we can consider the presheaf of categories

$$\underline{\mathcal{C}}\colon \mathcal{G}^{op} \to \mathsf{Add}\colon X \mapsto \underline{\mathsf{Rep}(X)} = \mathsf{Stab}(\mathsf{Rep}(X)).$$

If we evaluate this presheaf on certain special G-sets we obtain the stable categories we are interested in. In fact, given a subgroup $H \to G$ then there is a well-behaved equivalence $\underline{\mathcal{C}}(G/H) \simeq kH\text{-}\mathsf{Stab}$. Last time we explained the 'stack theorem' and unraveled it in more down-to-earth terms.

Theorem 4.1. *The presheaf $\underline{\mathcal{C}}$ is a stack on $\mathcal{G}$.*

Thus, for every cover $\mathcal{U} = (U \to X)$ the functor $\underline{\mathcal{C}}(X) \to \mathsf{Desc}_{\underline{\mathcal{C}}}(U)$ is an equivalence. For convenience let us recall that an object in the target category is a pair (W, γ) consisting of an object $W \in \underline{\mathcal{C}}(U)$ together with an isomorphism

$$\gamma\colon pr_2^*(W) \xrightarrow{\cong} pr_1^*(W)$$

in $\underline{\mathcal{C}}(U^{(2)})$. This isomorphism has to satisfy the following cocycle condition on $U^{(3)}$:

$$pr_{13}^*(\gamma) = pr_{12}^*(\gamma) \circ pr_{23}^*(\gamma)$$

There is a certain analogy between line bundles (as studied in algebraic geometry) and endotrivial modules (as studied in representation theory). From a conceptional perspective, both notions precisely encode the dualizable objects with respect to certain monoidal structures. In algebraic geometry, Čech cohomology is a convenient tool for studying line bundles. The aim of this lecture is to also use Čech cohomology to learn something about endotrivial modules.

Let us begin by recalling the definition of the Čech complex. For this purpose, let us consider a presheaf of abelian groups

$$F\colon \mathcal{G}^{op} \to \mathsf{Ab}$$

and let $\mathcal{U} = (U \to X)$ be a cover in $\mathcal{G}$ (e.g., $G/P \to G/G$ with P p-Sylow). Then we can consider the following sequence of homomorphisms of abelian groups:

$$F(U) \to F(U^{(2)}) \to F(U^{(3)}) \to \ldots$$

The homomorphisms are obtained as follows. Let $pr_i\colon U^{(n)} \to U^{(n-1)}$ be the projection away from the i-th factor. Then we obtain homomorphisms $pr_i^*\colon F(U^{(n-1)}) \to F(U^{(n)}))$ and we can hence form the alternating sum:

$$d = \Sigma_i (-1)^i pr_i^*\colon F(U^{(n-1)}) \to F(U^{(n)})$$

An easy calculation shows that this way we obtain a non-negative cochain complex.

Definition 4.2. The above cochain complex $\check{\mathrm{C}}^\bullet(\mathcal{U}; F)$ with $\check{\mathrm{C}}^n(\mathcal{U}; F) = F(U^{(n+1)})$ is the **Čech complex** associated to the presheaf F with respect to the cover $\mathcal{U}$. Its cohomology

$$\check{\mathrm{H}}^n(\mathcal{U}; F) = H^n(\check{\mathrm{C}}^\bullet(\mathcal{U}; F)), \quad n \geq 0,$$

is the n-th **Čech cohomology** of the presheaf F with respect to the cover $\mathcal{U}$.

Here, we will mainly be interested in two such presheaves. For the first one, let us recall that associated to $X \in \mathcal{G}$ there is the associated **trivial representation** $\mathbb{I}$ over k:

$$\mathbb{I}_x = k,\ x \in X, \qquad \text{and} \qquad \mathbb{I}_g = id,\ g \in G$$

The first presheaf is obtained by forming stable automorphisms of these trivial representations.

Example 4.3. The assignment $X \mapsto \mathrm{Aut}_{\underline{\mathcal{C}}(X)}(\mathbb{I})$ defines a presheaf of abelian groups on $\mathcal{G}$:

$$\mathbb{G}_m \colon \mathcal{G}^{op} \to \mathsf{Ab}$$

Exercise 4.4. Verify the details of the example.

We can describe the value of this presheaf on orbits more explicitly. For $X = G/H$ we have that $\mathbb{G}_m(G/H) = 1$ if p does not divide the order of H. In fact, the stable category is trivial in that case. If p does divide the order of H then we obtain $\mathbb{G}_m(G/H) = k^\times$.

It turns out that the presheaf $\mathbb{G}_m$ is actually a sheaf with respect to the sipp-topology. Using this we hence obtain that the 0-th Čech cohomology group is given by global sections.

The second presheaf we are interested in is related to the formation of Picard groups. Recall that given a symmetric monoidal category $(\mathcal{D}, \otimes)$ which only has a set of isomorphism classes of $\otimes$-invertible objects then this set forms an abelian group. This group is denoted by $\mathsf{Pic}(\mathcal{D})$ and is called the **Picard group** of $(\mathcal{D}, \otimes)$.

Example 4.5. The assignment $X \mapsto \underline{\mathsf{Pic}}(X) = \mathsf{Pic}(\underline{\mathcal{C}}(X))$ defines a presheaf of abelian groups on $\mathcal{G}$:

$$\underline{\mathsf{Pic}} \colon \mathcal{G}^{op} \to \mathsf{Ab}$$

This is the presheaf of **stable Picard groups**.

Exercise 4.6. Verify the details of this example by carrying out the following steps:

(1) Given a G-set X then the category $\mathsf{Rep}(X)$ of representations can be endowed with a symmetric monoidal structure induced form the usual tensor product of modules.
(2) This tensor product induces a tensor product at the level of stable categories $\underline{\mathcal{C}}(X)$.
(3) A morphism of G-sets induces a morphisms at the level of stable Picard groups. From the description of this morphism the functoriality should be immediate.

Theorem 4.7. *Given a sipp-cover $\mathcal{U} = (\alpha \colon U \to X)$ of $X \in \mathcal{G}$ then there is a natural isomorphism:*

$$\check{\mathrm{H}}^1(\mathcal{U}; \mathbb{G}_m) \cong \mathrm{Ker}\big(\underline{\mathsf{Pic}}(X) \xrightarrow{\alpha^*} \underline{\mathsf{Pic}}(U)\big)$$

We will not give a proof of the theorem here but only mention that it is formal in the sense that nothing special about the sipp-topology is used.

For the remainder of this lecture let us consider the special case of the sipp cover

$$\mathcal{U} = (G/P \to G/G = *)$$

where P is the p-Sylow subgroup of G. Our preferred equivalence $\underline{\mathcal{C}}(G/H) \simeq \mathcal{C}(H)$ induces an isomorphism

$$\underline{\mathsf{Pic}}(G/H) = \mathsf{Pic}(\underline{\mathcal{C}}(G/H)) \quad \cong \quad \mathsf{Pic}(\mathcal{C}(H)) = T(H)$$

where $T(H)$ denotes the group of **endotrivial modules**. Neither of the two Pic-groups in the theorem is an $\check{\mathrm{H}}^1$, but the kernel is.

The point of the theorem is that Carlson-Thévenaz have classified endotrivial modules on p-groups. This was a highly non-trivial task and is the subject of the two papers [CT04, CT05]. As an upshot we obtain that $\check{\mathrm{H}}^1(\mathcal{U}; \mathbb{G}_m)$ is in bijection to the set of set-theoretic maps $u \colon G \to k^\times$ which satisfy the following three conditions:

(1) $u(p) = 1$ if $p \in P$
(2) $u(g) = 1$ if $P^g \cap P = 1$
(3) $u(g_2 g_1) = u(g_2)u(g_1)$ if $P^{g_1 g_2} \cap P^{g_1} \cap P \neq 1$

Our final goal is to give a description of $\mathrm{Im}(T(G) \to T(P))$. There is a naive obstruction for an object $W \in T(P)$ to lie in the image. If $W \cong \mathrm{Res}^G_P(V)$ for some $V \in T(G)$ then necessarily

$$\mathrm{Res}^P_{P[g]}(W) \cong {}^g\mathrm{Res}^P_{P[g]}(W)$$

where again $P[g] = P^g \cap P$.

Recall that we are considering a sipp-cover $\mathcal{U} = (U = G/P \to X = G/G = *)$. Under the isomorphism $T(P) \cong \underline{\mathsf{Pic}}(U)$ the above naive obstruction translates into the condition on $W \in \underline{\mathsf{Pic}}(U)$ that it should satisfy:

$$pr_2^*(W) \cong pr_1^*(W) \quad \text{in} \quad \underline{\mathcal{C}}(U^{(2)})$$

So, let us *choose* arbitrary such isomorphisms $\gamma\colon pr_2^*(W) \overset{\cong}{\to} pr_1^*(W)$. Then we might wonder whether these isomorphisms satisfy

$$pr_{13}^*(\gamma) \overset{?}{=} pr_{12}^*(\gamma) \circ pr_{23}^*(\gamma),$$

i.e., do they satisfy a cocycle condition? This would allow us to extend W. There is no reason for this to hold, a priori. However, both sides of the equation are isomorphisms of an invertible object, thus they must differ uniquely by an isomorphism of the unit. Thus we obtain a unique

$$\zeta(W,\gamma) \in \mathrm{Aut}_{\underline{\mathcal{C}}(U^{(3)})}(\mathbb{I}) \quad \text{s.th} \quad \zeta(W,\gamma) = pr_{13}^*(\gamma)^{-1} \circ pr_{12}^*(\gamma) \circ pr_{23}^*(\gamma).$$

Note that $\mathrm{Aut}_{\underline{\mathcal{C}}(U^{(3)})}(\mathbb{I}) = \check{\mathrm{C}}^2(\mathcal{U}; \mathbb{G}_m)$ is a Čech cochain group and one checks that $d(\zeta(W,\gamma)) = 0$, i.e., that we have cycle.

Proposition 4.8. *The cohomology class* $[\zeta(W,\gamma)] \in \check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)$ *is independent of the choice of* γ.

One can check that that way we obtain a well-defined group homomorphism:

$$z\colon \check{\mathrm{H}}^0(\mathcal{U}; \underline{\mathsf{Pic}}) = \{W \in \underline{\mathsf{Pic}}(U) \mid pr_1^*(W) \cong pr_2^*(W)\} \to \check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)$$

It turns out that this invariant checks precisely whether W can be extended or not. More precisely, W can be extended if and only if it lies in the kernel of z. Thus we obtain the following theorem.

Theorem 4.9. *In the notation established above there is an isomorphism:*

$$\mathrm{Im}(T(G) \to T(P)) \quad \cong \quad \mathrm{Ker}\big(z\colon \check{\mathrm{H}}^0(\mathcal{U}; \underline{\mathsf{Pic}}) \to \check{\mathrm{H}}^2(\mathcal{U}; \mathbb{G}_m)\big)$$

For more comments and details see [Bal12].

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