PROBLEM SESSION
DESCENT TECHNIQUES IN MODULAR REPRESENTATION THEORY

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All rings are associative and unital. All modules are left modules.

1. LECTURE

Let R be a commutative ring.

Problem 1.1 (Separable algebras)**.** Let $R \to A$ be an algebra over the commutative ring R. Show that the following are equivalent:

(1) A is projective as A^e-module, where $A^e := A \otimes_R A^{\mathrm{op}}$ denotes the enveloping algebra, and the action is $(a \otimes a') \cdot x := axa'$.
(2) There exists an element $c \in A \otimes_R A$ such that $\mu(c) = 1$ (where $\mu\colon A \otimes_R A \to A$ is multiplication) and $(a \otimes 1) \cdot c = c \cdot (1 \otimes a)$ for all $a \in A$.
(3) The multiplication map $\mu\colon A \otimes_R A \to A$ has a section σ which is A-bilinear.

Problem 1.2. Suppose that $R = Rs_1 + \ldots + Rs_n$, i.e., that $\mathrm{Spec}(R) = D(s_1) \cup \cdots \cup D(s_n)$, where $D(s) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid s \notin \mathfrak{p}\}$ is the Zariski open subset associated to the element $s \in R$. Let $A := R_{s_1} \times \cdots \times R_{s_n}$, where $R_s := R[\frac{1}{s}] := \{\frac{r}{s^n} \mid r \in R, n \in \mathbb{N}\}/(\frac{r}{s^n} \sim \frac{t}{s^m} :\Leftrightarrow \exists \ell \in \mathbb{N} \text{ with } rs^{m+\ell} = ts^{n+\ell})$ is the commutative ring obtained from R by formally inverting s. "Expand" descent theory for A over R. Show that A satisfies descent.

Let $A_H^G = (k(G/H), \mu, \eta)$ be as defined in the lecture.

Problem 1.3. Show that A_H^G is a ring object and that it is commutative and separable (!).

Problem 1.4. Show that for a *separable* ring object (or monad) A, every A-module is "projective", i.e., a direct summand of a free module. In other words, show that the inclusion functor $A\text{-Free}_{\mathcal{C}} \hookrightarrow A\text{-Mod}_{\mathcal{C}}$ is "$\oplus$-dense".

Problem 1.5 (The idempotent completion)**.** Define the idempotent completion $\mathcal{A}^\natural$ of an additive category $\mathcal{A}$ and study its basic properties:

(1) $\mathcal{A}^\natural$ can be defined by the following construction, due to Karoubi. Its objects are pairs (x, e), where x is an object of $\mathcal{A}$ and $e =$

$e^2\colon x \to x$ an idempotent morphism. The Hom sets are the subgroups

$$\mathrm{Hom}_{\mathcal{A}^\natural}((x,e),(y,f)) := f \circ \mathrm{Hom}_{\mathcal{A}}(x,y) \circ e = \{\varphi \mid f\varphi = \varphi = \varphi e\}$$

of $\mathrm{Hom}_{\mathcal{A}}(x,y)$. (That is: show that this yields a well-defined idempotent complete additive category $\mathcal{A}^\natural$).

(2) The assignment $x \mapsto (x, 1_x)$ extends to a fully faithful additive functor $\iota_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}^\natural$.

(3) The assignment $\mathcal{A} \mapsto \mathcal{A}^\natural$ extends to a functor $(-)^\natural\colon \mathrm{Add} \to \mathrm{Add}_{ic}$ from the category of (small) additive categories to that of (small) idempotent complete additive categories.

(4) The idempotent completion is characterized, up to equivalence, by the following (2-)universal property: for every idempotent complete additive category $\mathcal{B}$, pre-composition with $\iota_{\mathcal{A}}$ induces an equivalence of categories

$$(\iota_{\mathcal{A}})^*\colon \mathrm{Fun}_{\mathrm{add}}(\mathcal{A}^\natural, \mathcal{B}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathcal{A}, \mathcal{B})\,.$$

(here $\mathrm{Fun}_{\mathrm{add}}$ denotes the category of additive functors and natural transformations between them).

(5) $(-)^\natural$ is almost (but not quite!) left adjoint to $\mathrm{Add}_{ic} \overset{\text{forget}}{\to} \mathrm{Add}$.

Let $H \leqslant G$ be finite groups.

Problem 1.6. When is the unit $\eta\colon k \to A^G_H$ retracted in kG-Mod? Say, for k a field of characteristic $p > 0$. What about in kG-Stab?

Problem 1.7. Let M be the monad associated with the adjunction

$$\begin{array}{c} kG\text{-Mod} \\ \mathrm{Res}^G_H \big\downarrow \big\uparrow \mathrm{Ind}^G_H \ \ (\text{as } \textit{right} \text{ adjoint!}) \\ kH\text{-Mod} \end{array}$$

(i.e.: $M = \mathrm{Ind}^G_H \circ \mathrm{Res}^G_H$, η = unit of adjunction, $\mu = \mathrm{Ind}^G_H\, \varepsilon\, \mathrm{Res}^G_H$). Check that $\mu\colon M^2 \to M$ identifies with the product $\mu\colon A^G_H \otimes_k A^G_H \to A^G_H$ (as defined in the lecture) under the usual "Frobenius" isomorphism $\mathrm{Ind} \circ \mathrm{Res} \cong A^G_H \otimes_k (-)$ given by

$$kG \otimes_{kH} \mathrm{Res}^G_H V \xrightarrow{\sim} k(G/H) \otimes_k V \quad , \quad g \otimes v \mapsto [g] \otimes gv$$

for all $V \in kG$-Mod.

Problem 1.8 (Orbit decomposition of pullbacks)**.** Let $K_1, K_2 \leqslant H \leqslant G$ be finite groups. Prove the Mackey formula for pullbacks, i.e., show that every choice of a full set $S \subset H$ of representatives for $K_1 \backslash H / K_2$ defines an isomorphism of left G-sets

$$\coprod_{t \in S} G/(t^{-1}K_1 t \cap K_2) \xrightarrow{\sim} (G/K_1) \times_{G/H} (G/K_2)$$

via the map $[g] \mapsto ([gt^{-1}], [g])$.