PROBLEM SESSION
DESCENT TECHNIQUES IN MODULAR REPRESENTATION THEORY

PAUL BALMER

2. LECTURE

Problem 2.1. Let X be a topological space and let $\mathcal{G} := \mathcal{O}(X)$ be the category with objects the open subsets of X and with morphisms the inclusions between them:

$$\mathrm{Hom}_{\mathcal{O}(X)}(U, V) = \begin{cases} \{U \to V\} & \text{if } U \subseteq V \\ \emptyset & \text{if } U \not\subseteq V. \end{cases}$$

Define a family of morphisms $\{U_i \to U\}_{i\in I}$ in $\mathcal{O}(X)$ to be a *covering* if their domains cover their common target, in the usual sense: $U = \bigcup_{i=I} U_i$. Verify that these covering families form a Grothendieck topology on $\mathcal{O}(X)$:

(1) $\{U \overset{\sim}{\to} V\}$ is a covering for every isomorphism $U \overset{\sim}{\to} V$.
(2) The pullbacks $\{U_i \times_U V \to V\}_{i\in I}$ (taken in $\mathcal{O}(X)$!) of a covering $\{U_i \to U\}_{i\in I}$ along any morphism $V \to U$ form a covering.
(3) If $\{U_i \to V\}_{i\in I}$ and $\{V_{ij} \to U_i\}_{j\in J_i}$ $(i \in I)$ are coverings then the compositions $\{V_{ij} \to U\}_{i\in I, j\in J_i}$ form a covering.

Now verify that a presheaf on X is the same as a functor $\mathcal{O}(X)^{\mathrm{op}} \to$ Set, and that a sheaf on X is the same as a sheaf on $\mathcal{O}(X)$, as in the general definition for sites with pullbacks (ask!).

Problem 2.2. Finish the verification that the *sipp topology* on $\mathcal{G} :=$ G-Set is indeed a Grothendieck topology (cf. the second lecture).

Problem 2.3. Show that $X \in G$-Set is *local* in the sipp topology (meaning: for every sipp covering family $\{Y_i \to X\}_i$ one of the $Y_i \to X$ admits a section) if and only if $X \cong G/H$ with H a p-group. Thus H is a p-Sylow of G iff $G/H \to G/G$ is a sipp covering by a local object.

Problem 2.4 (Action groupoid)**.** Let X be a left G-set. Verify the soundness of the following definition: the *action groupoid* $G \ltimes X$ is the category (in fact, a groupoid) with the elements of X as objects and with morphisms given by

$$\mathrm{Hom}_{G\ltimes X}(x, x') := \{g \in G \mid gx = x'\}.$$

Show that $\mathrm{Rep}_k(X)$ (as defined in the lecture) is the category $k\text{-Mod}^{G\ltimes X}$ of (covariant) functors from $G \ltimes X$ to k-Mod.

Problem 2.5. Let $H \leqslant G$. Show that the group H (seen as the category with one object $*$ whose automorphism group $\mathrm{Aut}(*)$ is H) is equivalent to the action groupoid $G \ltimes G/H$, via the functor $\iota_H \colon * \mapsto [1]$. Deduce an equivalence $(\iota_H)^* \colon \mathrm{Rep}(G/H) \xrightarrow{\sim} kH\text{-Mod}$, and observe that this is the equivalence mentioned in the course.

Problem 2.6. Recall (or ask!) what the derived category and the stable categories are.

Problem 2.7. For $K \leqslant H \leqslant G$, show that the following diagram commutes:

$$\begin{array}{ccc} \underline{\mathcal{C}}(G/H) & \xrightarrow[\sim]{\iota_H^*} & \mathcal{C}(H) \\ {\scriptstyle \alpha^*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Res}^H_K} \\ \underline{\mathcal{C}}(G/K) & \xrightarrow[\sim]{\iota_K^*} & \mathcal{C}(K) \end{array}$$

where $\alpha \colon G/K \to G/H$, $[g] \mapsto [g]$, is the G-map induced by $K \subseteq H$. (The notations $\mathcal{C}(\cdots)$, $\underline{\mathcal{C}}(\cdots)$ and $(\cdots)^*$ are as in the lecture.)

Problem 2.8. For ${}^gK := gKg^{-1} \leqslant H \leqslant G$, show that the square

$$\begin{array}{ccc} \underline{\mathcal{C}}(G/H) & \xrightarrow[\sim]{\iota_H^*} & \mathcal{C}(H) \\ {\scriptstyle \beta_g^*}\big\downarrow & \overset{\omega^{(g)}}{\nearrow} & \big\downarrow{\scriptstyle {}^g\mathrm{Res}^H_K} \\ \underline{\mathcal{C}}(G/K) & \xrightarrow[\sim]{\iota_K^*} & \mathcal{C}(K) \end{array}$$

commutes ("only") up to an isomorphism $\omega^{(g)} \colon i_K^* \circ \beta_g^* \xrightarrow{\sim} {}^g\mathrm{Res}^H_K \circ i_H^*$ of functors, as indicated by the diagonal arrow. Here β_g denotes the G-map $G/K \to G/H$ given by $[x] \mapsto [xg^{-1}]$, and ${}^g\mathrm{Res}^H_K(V)$ is the "twisted restriction" of the kH-module V, with K-action $k \cdot v := {}^gkv$ (for $k \in K, v \in V$). If ${}^{g_1}K_1 \leqslant K_2$ and ${}^{g_2}K_2 \leqslant H$, explain the precise relation between the isomorphisms associated to ${}^{g_1}K_1 \leqslant H$, ${}^{g_2}K_2 \leqslant K_1$ and ${}^{g_1g_2}K_2 \leqslant H$ (draw a picture!).