November 23, 2016

Announcements

- This week 8.3 (Center of Mass)
- Homework # 9A (Center of Mass) & 9B (Separable differential equations) Due Wednesday, November 30, 11:00pm
- Happy Thanksgiving!
- Midterm #2: 232 exams, median = 35 and mean = 35.05
  - Requests for corrections to arithmetic errors in the midterm grade must be made in writing and handed in, with your exam, to either me or your TA Today

Today

- Continue & finish: 8.3 Center of Mass
  (Note: We skip Hydrostatic Pressure and Force)

Center of Mass of a Lamina

Consider a lamina with uniform density that occupies a region R. The center of mass of R is the centroid of R. To locate the centroid we need to follow 3 physical principles:

1. The symmetry principle: if R is symmetric about a line l the centroid of R lies on l.
2. Moments should be defined so that if the entire mass of the region is concentrated at the centroid, then its moments remain unchanged.
3. The moment of the union of 2 non-overlapping regions should be the sum of the moments of the individual regions.

Let $R = \{(x, y) : a \leq x \leq b,\ 0 \leq y \leq f(x)\}$ where f is a continuous function on $[a, b]$. R has uniform density ρ.

$$M_y = \rho \int_a^b x f(x)\, dx,$$

$$M_x = \frac{\rho}{2} \int_a^b [f(x)]^2\ dx.$$

If A denotes the area of R, then the center of mass of R is located at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A} \int_a^b x f(x)\, dx,$$

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)]^2\ dx.$$

Recall that $A = \displaystyle\int_a^b f(x)\, dx$

A word about Moments

Students often find the topic of Moments confusing, in particular as it does not agree with the notion of "Moments of inertia" that they may have learned about in Physics class.

The **Moments** of the lamina about the y-axis and x-axis that we have just defined were derived using the **distance** to the axis and are closely connected to the center of mass. In classical physics literature these would be called "first moments" (of mass).

The **Moments of inertia** (also called the "second moments" of mass) are derived using the **square of the distance** to the axes. These are related to the notion of kinetic energy when studying rotational motion.

Let $R = \{(x, y) : a \le x \le b,\ g(x) \le y \le f(x)\}$ where f and g are continuous functions on $[a, b]$ then the center of mass of R is located at $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{A} \int_a^b x\,[f(x) - g(x)]\ dx,$$

and

$$\bar{y} = \frac{1}{A} \int_a^b \frac{1}{2} \left([f(x)]^2 - [g(x)]^2 \right)\ dx.$$

Example:

Find the centroid of the region bounded by

$$y = x + 2 \qquad and \qquad y = x^2.$$

Theorem of Pappus

Let R be a planar region that lies entirely on one side of a line l in the plane. If R is rotated about l, then the volume of the resulting solid is the product of the area A of R and the distance d traveled by the centroid of R.

Example: A torus is formed by rotating a circle of radius r about a line in a plane of the circle that is a distance $R > r$ from the center of the circle. Find the volume of the torus.