October 4, 2004

Announcements

- This Week: §5.2, 5.3 and 5.4
- You need to have the "Math 125 course pak".
  ($8.50 from the UW Copy Center in
  Communications (CMU) B042.)
  Always bring this to your Tu & Th sections.
- Congratulations on making good use of the
  Math 125 C&D email list!
- Homework #1 (Week 1 Problems)
  Due Tomorrow
  (Covers §4.10, 5.1 and 5.2; see web for assignment)
  Collected: Thursday, October 7
- Quiz # 1 Thursday, October 7
  (Covers §4.10, 5.1 and 5.2, Week 1 Problems).

Today

- §5.2 The Definite Integral: definition and properties.

When solving the area problem we encountered, **Riemann sums**, which are expressions of the form

$$\sum_{i=1}^{n} f(x_i^*)\Delta x =$$

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x.$$

We also studied their limits, i.e.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x =$$

$$\lim_{n\to\infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x \cdots + f(x_n^*)\Delta x].$$

Definition: Let f be a continuous function defined on the interval $[a, b]$. Divide the interval $[a, b]$ into n-subintervals of equal width

$$\Delta x = \frac{b-a}{n}.$$

Let

$$x_0 = a,\ x_n = b,\ x_{i+1} = x_i + \Delta x$$

Let $x_i^* \in [x_{i-1}, x_i]$ be sample points. The **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Proposition:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Proof (of the first): Note that

$$\begin{array}{lcccccccc} \sum_{i=1}^{n} i & = & 1 & + & 2 & + & \cdots & + & n \\ \sum_{i=1}^{n} i & = & n & + & (n-1) & + & \cdots & + & 1 \end{array}$$

Adding we have

$$2\sum_{i=1}^{n} i = (n+1)+(n+1)+\cdots+(n+1)$$

$$= n(n+1).$$

To compute $\sum_{i=1}^{n} i^2$ we use a *telescoping sum*. Note that

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1$$

and

$$\begin{aligned}\sum_{i=1}^{n} [(i+1)^3 - i^3] &= [2^3 - 1^3] + [3^3 - 2^3] \\ &\quad + \cdots + [(n+1)^3 - n^3] \\ &= (n+1)^3 - 1 \\ &= n^3 + 3n^2 + 3n.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{i=1}^{n} [(i+1)^3 - i^3] &= \sum_{i=1}^{n} 3i^2 + 3i + 1 \\ &= 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \\ &= 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n.\end{aligned}$$

Combining the 2 equalities above we have

$$
\begin{aligned}
3\sum_{i=1}^{n} i^2 &= n^3 + 3n^2 - 3\frac{n(n+1)}{2} + 2n \\
&= n^3 + \frac{3}{2}n^2 + \frac{1}{2}n \\
&= \frac{2n^3 + 3n^2 + n}{2} \\
&= \frac{n(n+1)(2n+1)}{2}
\end{aligned}
$$

Properties of sums

- $\sum_{i=1}^{n} \lambda = n\lambda$

- $\sum_{i=1}^{n} \lambda a_i = \lambda \sum_{i=1}^{n} a_i$

- $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$

- $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$

Properties of the definite integral

If $c \in [a, b]$ and λ is a constant then

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b \lambda\,dx = \lambda(b - a)$
- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
- $\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$

Comparison properties for the integral

- If $f(x) \geq 0$ for $x \in [a, b]$, then

$$\int_a^b f(x)\, dx \geq 0$$

- If $f(x) \geq g(x)$ for $x \in [a, b]$, then

$$\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$$

- If $m \leq f(x) \leq M$ for for $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a).$$